SPECTRAL THEORY OF A MATHEMATICAL MODEL IN QUANTUM FIELD THEORY FOR ANY SPIN

JEAN-CLAUDE GUILLOT

Abstract. In this paper we use the formalism of S.Weinberg in order to construct a mathematical model based on the weak decay of hadrons and nuclei. In particular we consider a model which generalizes the weak decay the nucleus of the cobalt. We associate with this model a Hamiltonian with cutoffs in a Fock space. The Hamiltonian is self-adjoint and has an unique ground state. By using the commutator theory we get a limiting absorption principle from which we deduce that the spectrum of the Hamiltonian is absolutely continuous above the energy of the ground state and below the first threshold.

1. Introduction

This article initiates the study of mathematical models based on the Quantum Field Theory without any restriction concerning the spins of the involved particles.

Precisely, in this paper, we study a mathematical model which generalizes the weak decay of the nucleus $^{60}_{27}Co$ of spin 5 into the nucleus $^{60}_{28}Ni^{*}$ of spin 4, one electron and the antineutrino associated to the electron. This experiment by C.S.Wu and her collaborators showed that parity conservation is violated in the β decay of atomic nuclei. See [26]. The same approach can be applied to many examples of weak decays of hadrons and nuclei with both Fermi and Gamow-Teller transitions. See [37] and [26].

The mathematical model is based on the construction of free causal fields associated with two massive bosons of spins j_1 and j_2 respectively, a massive fermion of spin j_3 and a massless fermion of helicity $-j_4$ which is the antiparticle of a massless fermion of helicity j_4. These free causal fields are constructed according to the formalism described by S.Weinberg in [46, 47, 48, 49, 50, 51](see also [42],[27]).

This construction depends on the unitary irreducible representations of the Poincaré group for massive and massless particles and on the finite dimensional representations of $SL(2, \mathbb{C})$. Relativistic covariance laws and microscopic causality conditions determine unique free causal fields up to over-all scales. Note that in this paper we only consider fields associated with irreducible finite dimensional representations of $SL(2, \mathbb{C})$ because we are only concerned with a weak decay for which parity is not conserved.

As it is well known, the construction of the unitary irreducible representations of the Poincaré group for massive particles of any spin and for massless ones with any finite helicity is based on the theory of E.P.Wigner and G.W.Mackey. We choose the realizations of the unitary irreducible representations of the Poincaré group given by E.P.Wigner because they are important from the physical point of view and because they allow a clear distinction between the canonical and helicity formalisms.

The interaction between particles is the one given by S.Weinberg in [51, chap 5]. As for the weak interactions we do not suppose that the interaction commutes with space inversion.

After introducing convenient cutoffs for the associated Hamiltonian the mathematical method used to study the spectral properties of the Hamiltonian is based on the one applied to a mathematical model associated with the weak decay of the intermediate vector bosons $W^{\pm}$ into the family of leptons which has been recently developed by [9, 3]. The existence of a ground state and the proof that the spectrum of the Hamiltonian is absolutely continuous above the energy of the ground state and below the first threshold for a sufficiently small coupling constant are our main results. Our methods are largely taken from [4, 19, 13] and are based on [35, 2, 41, 21, 25, 23]. No infrared regularization is assumed.

In the framework of non-relativistic QED similar results have been successfully obtained for the massless Pauli-Fierz models (see [5, 6, 7, 22, 19, 20, 12] and references therein).

For other mathematical models in Quantum Field Theory see, for example, [1, 8, 24] and for string-localized quantum fields see [38] and references therein.

The paper is organized as follows. In the next section we recall the realizations of the unitary irreducible representations of the Poincaré group obtained by E.P.Wigner. In section 3 we first introduce the Fock spaces and the creation and annihilation operators with their usual commutation or anticommutation relations for massive particles. We then recall the construction of the finite dimensional irreducible representations of $SL(2, \mathbb{C})$ and we give a very detailed review of the construction of free causal fields for a massive particle of any spin following the formalism of S.Weinberg and associated with a finite dimensional irreducible representation of $SL(2, \mathbb{C})$. Similarly in section 4 we recall the construction of free causal fields for massless particles of any finite helicity according to the same formalism as for the massive particles. In section 5 we describe the model for the weak decay of a massive boson into a massive boson, a massive fermion and a massless fermion which can be an antineutrino generalizing the model for the weak decay of the nucleus ${}^{60}_{27}Co$. In section 6 we associate a self-adjoint Hamiltonian in a Fock space with this model and in section 7 we finally give our main results concerning the spectrum of the self-adjoint Hamiltonian.

2. The Poincaré Group

Let us recall that the Minkowski space is $\mathbb{R}^4$ equipped with the Lorentz inner product which is the bilinear form L defined by

$$L(x, y) = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3 \ . \tag{2.1}$$

$x^0 = ct$, where t is the time coordinate and c the speed of light. (x^1, x^2, x^3) is a set of cartesian coordinates on $\mathbb{R}^3$.

From now on we choose units such that $c = \hbar = 1$.

The Lorentz form L is associated with the metric

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\mu = g^{\mu\nu}dx_\mu dx_\mu \\ g_{\mu\nu} &= g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \ . \end{aligned} \tag{2.2}$$

with $\mu, \nu = 0, 1, 2, 3.$ and where we denote by x^μ (resp.x_μ) the vector (x^0, x^1, x^2, x^3) (resp.(x_0, x_1, x_2, x_3)). We use the Einstein summation convention.

A point $x \in \mathbb{R}^4$ may be written as $(x^0, \mathbf{x})$ where $\mathbf{x} = (x^1, x^2, x^3)$.

Note that $x^\mu = g^{\mu\nu} x_\nu$ and $x_\mu = g_{\mu\nu} x^\nu$.

The restricted Lorentz group or proper Lorentz group, denoted by $\mathcal{L}$, is the group of all linear real transformations $\Lambda = (\Lambda^\mu{}_\nu)$ such that

$$L(\Lambda x, \Lambda y) = L(x, y) \tag{2.3}$$

$$\det \Lambda = 1 \tag{2.4}$$

$$\Lambda_0^0 \geq 1 \ . \tag{2.5}$$

The rotation group $SO(3)$ is the orthogonal subgroup of $\mathcal{L}$ that fixes the point $(1, 0, 0, 0)$.

The inhomogeneous Lorentz group is the group of transformations of $\mathbb{R}^4$ generated by $\mathcal{L}$ and the group of translations isomorphic to $\mathbb{R}^4$ itself. The inhomogeneous group is the semi-direct product of $\mathcal{L}$ and $\mathbb{R}^4$, denoted by $\mathcal{L} \ltimes \mathbb{R}^4$, with group law given by

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2) \ . \tag{2.6}$$

where $\Lambda_j \in \mathcal{L}$ and $a_j \in \mathbb{R}^4$, $j = 1, 2$.

The action of (Λ, a) on $\mathbb{R}^4$ is

$$(\Lambda, a)x = \Lambda x + a \ . \tag{2.7}$$

According to E.P.Wigner and V.Bargmann (see [10],[43], [53] and [54]), in relativistic quantum mechanics, every projective representation of the inhomogeneous Lorentz group has a lift to an unitary representation of the universal covering group of the inhomogeneous Lorentz group. It is well known that the universal covering group of the inhomogeneous Lorentz group is the semi-direct product of $SL(2, \mathbb{C})$ and of $\mathbb{R}^4$ with the following law group

$$(A, a)(B, b) = (AB, a + \Lambda(A)b) \ . \tag{2.8}$$

Recall that $SL(2, \mathbb{C})$ is the group of the 2×2 complex matrices A such that $\det(A) = 1$. $\Lambda(A)$ is the image of A in the Lorentz group by the double covering of $\mathcal{L}$ by $SL(2, \mathbb{C})$ and is defined below.

The usual three Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$ together with σ_0, the unit 2×2 matrix on $\mathbb{C}^2$, generate the 2×2 hermitian matrices. We set $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. We identify $\mathbb{R}^4$ with a hermitian matrix by the map

$$p = (p^0, \mathbf{p}) \to p^\mu \sigma_\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} \tag{2.9}$$

where $\mu = 0, 1, 2, 3$.

Every $A \in SL(2, \mathbb{C})$ acts on $p^\mu \sigma_\mu$ by

$$p^\mu \sigma_\mu \to A(p^\mu \sigma_\mu)A^* \tag{2.10}$$

and there exists $\Lambda(A) \in \mathcal{L}$ such that

$$(\Lambda(A)p)^\mu \sigma_\mu = A(p^\mu \sigma_\mu)A^* \tag{2.11}$$

with

$$\Lambda(A)^\mu{}_\nu = \frac{1}{2} Tr(\sigma_\mu A \sigma_\nu A^*) \tag{2.12}$$

The map $A \to \Lambda(A)$ is a double covering of $\mathcal{L}$ by $SL(2, \mathbb{C})$ such that $\Lambda(A) = \Lambda(-A)$.

From now on we call Poincaré group the universal covering group of the inhomogeneous Lorentz group with the law group defined by (2.8). The Poincaré group is denoted by $\mathcal{P}$.

The subgroup $SU(2)$ of 2×2 unitary matrices of $SL(2,\mathbb{C})$ is the universal covering group of $SO(3)$. The covering map is the restriction of the one of $SL(2,\mathbb{C})$ to $SU(2)$.

Let $R(\mathbf{n},\theta)$ be the rotation of axis $\mathbf{n}$ and angle θ in $\mathbb{R}^4$. We have

$$\begin{aligned} \mathbf{x}' &= (\cos\theta)\mathbf{x} + (1-\cos\theta)(\mathbf{x}.\mathbf{n})\mathbf{n} + \sin\theta(\mathbf{n}\wedge\mathbf{x}) \\ x'^{\,0} &= x^0 \end{aligned} \tag{2.13}$$

where $\mathbf{x}.\mathbf{n} = x^1n^1 + x^2n^2 + x^3n^3$.

The following 2×2 matrix

$$A(\mathbf{n},\theta) = \cos\frac{\theta}{2}\sigma_0 - i\sin\frac{\theta}{2}(\mathbf{n}.\boldsymbol{\sigma}) = \mathrm{e}^{-i\theta\mathbf{n}.\frac{\boldsymbol{\sigma}}{2}} \tag{2.14}$$

is associated with $R(\mathbf{n},\theta)$ by the double covering of $\mathcal{L}$ by $SL(2,\mathbb{C})$. Thus $R(\mathbf{n},\theta) = \Lambda(A(\mathbf{n},\theta)$.

Let $L(\chi,\mathbf{m})$ be the pure Lorentz transformation in $\mathcal{L}$ in the direction $\mathbf{m} = (m^1,m^2,m^3)$ and with rapidity $v=\tanh\chi$ in the Minkowski space. We have

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - (1-\cosh\chi)(\mathbf{x}.\mathbf{m})\mathbf{m} + x^0(\sinh\chi)\mathbf{m} \\ x'^{\,0} &= (\cosh\chi)x^0 + (\mathbf{x}.\mathbf{m})\sinh\chi \end{aligned} \tag{2.15}$$

where $\mathbf{x}.\mathbf{m} = x^1m^1 + x^2m^2 + x^3m^3$.

In $SL(2,\mathbb{C})$ the following 2×2 matrix

$$A(\chi,\mathbf{m}) = \cosh\frac{\chi}{2}\sigma_0 + \sinh\frac{\chi}{2}(\mathbf{m}.\boldsymbol{\sigma}) = \mathrm{e}^{\chi\mathbf{m}.\frac{\boldsymbol{\sigma}}{2}} \tag{2.16}$$

is associated with $L(\chi,\mathbf{m})$ by the double covering of $\mathcal{L}$ by $SL(2,\mathbb{C})$. Thus $L(\chi,\mathbf{m}) = \Lambda(A(\chi,\mathbf{m})$.

For $R(\mathbf{n},\theta)$,$L(\chi,\mathbf{m})$ and throughout this work we follow the active point of view of transformations. See, for example, [52].

Note that

$$\begin{aligned} A(\mathbf{n},\theta)A(\chi,\mathbf{m}) &= A(\chi,R(\mathbf{n},\theta)\mathbf{m})A(\mathbf{n},\theta) \\ A(\chi,\mathbf{m})A(\mathbf{n},\theta) &= A(\mathbf{n},\theta)A(\chi,R(\mathbf{n},\theta)^{-1}\mathbf{m}) \end{aligned} \tag{2.17}$$

In relativistic quantum mechanics elementary systems are associated with unitary irreducible representations of $\mathcal{P}$. From this point of view elementary particles are elementary systems (see [53]). It can be also necessary to introduce the extended Poincaré group by considering discrete transformations such as space-inversion and time-reversal.

The description of irreducible unitary representations of $\mathcal{P}$ has been first accomplished by E.P.Wigner (see [54]). It is now treated as an application of the work of G.W. Mackey using induced representations.Many articles and books have been devoted to this theory. We only mention some of them. See [45], [11], [43], [18] and references therein.

We still keep the realization of the physical irreducible unitary representations of the $\mathcal{P}$ by E.P.Wigner because they are associated with spectral representations of maximal sets of commuting observables as the momenta, the spins or the helicities which are fundamental in dealing with kinematical problems for elementary particles.

We have to consider two cases in physics. First, the case of a positive mass $m>0$ and a spin j, with j integer or half-integer, i.e.$j\in\mathbb{N}$ or $j\in\mathbb{N}+\frac{1}{2}$. Second,

the case of a mass $m = 0$ and a helicity $j \in \mathbb{Z}$ or $j \in \mathbb{Z} + \frac{1}{2}$ for which the spin is $| j |$. In both cases the energy has to be positive.

2.1. **Positive mass and spin j.**

Let Ω_m be the orbit corresponding to the mass $m > 0$. We have

$$\Omega_m = \{p \in \mathbb{R}^4;\, p_\mu p^\mu = m^2,\, p^0 > 0\} \tag{2.18}$$

Observe that $p \in \Omega_m$ if and only if $p = (\omega_{\mathbf{p}}, \mathbf{p})$ where $\omega_{\mathbf{p}} = \sqrt{| \mathbf{p} |^2 + m^2}$. Here $| \mathbf{p} |= \sqrt{p_1^2 + p_2^2 + p_3^2}$.

The Lorentz invariant measure on Ω_m is $\frac{\mathrm{d}^3\mathbf{p}}{2\omega_{\mathbf{p}}}$.

Set

$$k_m = (m, 0, 0, 0) \tag{2.19}$$

The little group of k_m is $SU(2)$ which determines the spin of the particle.

The unitary irreducible representations of $SU(2)$ are finite dimensional ones and they are well known . See, for example, [17], [39], [34], [36] and [44].

Let $D^j(.)$ be the unitary irreducible representation of $SU(2)$ defined on a Hilbert space of dimension $2j + 1$ that, for simplicity, we suppose to be $\mathbb{C}^{2j+1}$. The irreducible unitary representation of mass $m > 0$ and spin j is defined on the Hilbert space $L^2(\Omega_m, \frac{\mathrm{d}^3\mathbf{p}}{2\omega_{\mathbf{p}}}, \mathbb{C}^{2j+1})$ with the scalar product

$$(F, G) = \int_{\Omega_m} F(p).G(p)_{2j+1} \frac{\mathrm{d}^3\mathbf{p}}{2\omega_{\mathbf{p}}} \tag{2.20}$$

where $\{F, G\} \longmapsto F(p).G(p)_{2j+1}$ is the scalar product in $\mathbb{C}^{2j+1}$ which is linear with respect to G and anti-linear with respect to F.

The unitary irreducible representation of $\mathcal{P}$ of mass $m > 0$ and spin j depends on a field of transformations of the restricted Lorentz group $p \longmapsto \Lambda(A_p)$ such that, for every p,

$$\Lambda(A_p)k_m = p \tag{2.21}$$

Given the field $p \to \Lambda(A_p)$, the unitary irreducible representation of the $\mathcal{P}$ of mass $m > 0$ and spin j, denoted by $U^{[m,j]}(A, a)$, is then

$$(U^{[m,j]}(A, a)F)(p) = \mathrm{e}^{ia.p} D^j(A_p^{-1} A A_{\Lambda(A)^{-1}p}) F(\Lambda(A)^{-1}p)\ . \tag{2.22}$$

where $a.p = a_\mu p^\mu$ and $F \in L^2(\Omega_m, \frac{\mathrm{d}^3\mathbf{p}}{2\omega_{\mathbf{p}}}, \mathbb{C}^{2j+1})$.

In physics one considers two interesting examples of the field $p \to \Lambda(A_p)$.

2.1.1. *The canonical formalism.* In that case $\Lambda(A_p)$ is the pure Lorentz transformation in the direction $\frac{\mathbf{p}}{|\mathbf{p}|}$. We then have

$$\begin{aligned} \mathbf{p} &= (\sinh \chi) m \frac{\mathbf{p}}{|\mathbf{p}|} \\ p^0 &= (\cosh \chi) m \end{aligned} \tag{2.23}$$

This pure Lorentz transformation is associated with the following element of $SL(2, \mathbb{C})$, denoted by A_p^C, by the double covering of $\mathcal{L}$ by $SL(2, \mathbb{C})$:

$$\begin{aligned} A_p^C = \frac{1}{2}(\sqrt{\frac{p_0 + |\mathbf{p}|}{m}} + \sqrt{\frac{p_0 - |\mathbf{p}|}{m}}) \\ + \frac{1}{2}(\sqrt{\frac{p_0 + |\mathbf{p}|}{m}} - \sqrt{\frac{p_0 - |\mathbf{p}|}{m}})(\frac{\mathbf{p}.\boldsymbol{\sigma}}{|\mathbf{p}|})\ . \end{aligned} \tag{2.24}$$

An easy computation shows that

$$A_p^C = \frac{(m+p_0)\sigma_0 + \mathbf{p}.\boldsymbol{\sigma}}{\sqrt{2m(m+p_0)}} \ . \tag{2.25}$$

For the choice of A_p^C the corresponding formalism is called canonical. See [53].

2.1.2. *The helicity formalism.* In that case $\Lambda(A_p)$ is the product of a pure Lorentz transformation Λ^H such that

$$\Lambda^H k_m = (p^0, 0, 0, |\, \mathbf{p}\,|) \ . \tag{2.26}$$

and a rotation R^H of axis $\mathbf{k} \wedge \frac{\mathbf{p}}{|\mathbf{p}|}$ and angle $\theta = (\mathbf{k}, \frac{\mathbf{p}}{|\mathbf{p}|})$ where $\mathbf{k}$ is the unit vector of the third axis.

$R^h\Lambda^H$ is associated with the following element of $SL(2,\mathbb{C})$, denoted by A_p^H, by the double covering of $\mathcal{L}$ by $SL(2,\mathbb{C})$:

$$(\sqrt{2m(m+p_0)})A_p^H = \begin{pmatrix} \alpha(p)\mathrm{e}^{-i\frac{\varphi}{2}}\cos\frac{\theta}{2} & -\beta(p)\mathrm{e}^{-i\frac{\varphi}{2}}\sin\frac{\theta}{2} \\ \alpha(p)\mathrm{e}^{i\frac{\varphi}{2}}\sin\frac{\theta}{2} & \beta(p)\mathrm{e}^{i\frac{\varphi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \tag{2.27}$$

Here $\alpha(p) = (m + p^0 + |\mathbf{p}|)$ and $\beta(p) = (m + p^0 - |\mathbf{p}|)$. Furthermore θ (resp.φ) is the polar(resp.azimutal) angle of $\mathbf{p}$ with $0 \le \theta \le \pi$ (resp. $0 \le \varphi < 2\pi$).

The corresponding formalism is called the helicity one. See [33], [50], [29] and [28].

Note that A_p^H is defined up to a rotation of axis $\mathbf{k}$. For example we can also use the following 2×2 matrix for $(\sqrt{2m(m+p_0)})A_p^H$:

$$\begin{pmatrix} \alpha(p)\cos\frac{\theta}{2} & -\beta(p)\mathrm{e}^{-i\varphi}\sin\frac{\theta}{2} \\ \alpha(p)\mathrm{e}^{i\varphi}\sin\frac{\theta}{2} & \beta(p)\cos\frac{\theta}{2} \end{pmatrix} \tag{2.28}$$

See [28].

2.2. **Mass m=0 and helicity j.**

Let Ω be the light cone:

$$\Omega = \{p^\mu p_\mu = 0\,;\, p^0 > 0\} \ . \tag{2.29}$$

Set

$$k_0 = (1, 0, 0, 1) \tag{2.30}$$

The little group of k_0 is the spinorial group of the euclidean group in $\mathbb{R}^2$ which is the group of rigid motions in $\mathbb{R}^2$ denoted by E_2. E_2 is the set of motions $(R(\varphi), a)$ in $\mathbb{R}^2$ such that, for u , v and $a \in \mathbb{R}^2$,

$$u = (R(\varphi))v + a \ . \tag{2.31}$$

Here $R(\varphi)$ is a rotation of angle φ in $\mathbb{R}^2$ whose center is the origin 0.

The law group of E_2 is

$$(R(\varphi_1), a_1)(R(\varphi_2), a_2) = (R(\varphi_1+\varphi_2), a_1 + (R(\varphi_1)a_2) \ . \tag{2.32}$$

The spinorial group of E_2 is the following set of 2×2 matrices:

$$\{z, \varphi\} = \begin{pmatrix} \mathrm{e}^{-i\frac{\varphi}{2}} & z \\ 0 & \mathrm{e}^{i\frac{\varphi}{2}} \end{pmatrix} \tag{2.33}$$

where $z \in \mathbb{C}$ and $\varphi \in \mathbb{R}$ with the law group

$$\{z_1, \varphi_1\}\{z_2, \varphi_2\} = \{z_1\mathrm{e}^{i\frac{\varphi_2}{2}} + z_2\mathrm{e}^{-i\frac{\varphi_1}{2}}, \varphi_1 + \varphi_2\} \ . \tag{2.34}$$

The spinorial group of E_2 is a double covering of E_2. The $2 \to 1$ homomorphism of the spinorial group over E_2 is given by

$$\{z, \varphi\} \longrightarrow (R(\varphi), a(z\mathrm{e}^{i\frac{\varphi}{2}})) \ . \tag{2.35}$$

where

$$a(z\mathrm{e}^{i\frac{\varphi}{2}}) = (\mathrm{Re}(z\mathrm{e}^{i\frac{\varphi}{2}}), \mathrm{Im}(z\mathrm{e}^{\frac{i\varphi}{2}})) \in \mathbb{R}^2. \tag{2.36}$$

Note that $\{z, \varphi\}$ and $\{-z, \varphi + 2\pi\}$ correspond to the same element in E_2

The unitary irreducible representations of the spinorial group associated to a finite helicity are of dimension one. They are indexed by $j \in \mathbb{Z}$ or $j \in \mathbb{Z} + \frac{1}{2}$. They are given by

$$L^j(\{z, \varphi\}) = \mathrm{e}^{-ij\varphi} \tag{2.37}$$

j is the helicity and $|j|$ is the spin.

Remark 2.1. *The spinorial group of E_2 is isomorphic to the group generated by the following set of 2×2 matrices:*

$$\left[z, \varphi\right] = \begin{pmatrix} \mathrm{e}^{-i\frac{\varphi}{2}} & z\mathrm{e}^{i\frac{\varphi}{2}} \\ 0 & \mathrm{e}^{i\frac{\varphi}{2}} \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-i\frac{\varphi}{2}} & 0 \\ 0 & \mathrm{e}^{i\frac{\varphi}{2}} \end{pmatrix} \tag{2.38}$$

with the law group

$$\left[z_1, \varphi_1\right]\left[z_2, \varphi_2\right] = \left[z_1 + z_2\mathrm{e}^{-i\varphi_1}, \varphi_1 + \varphi_2\right] \ . \tag{2.39}$$

The unitary irreducible representation of $\mathcal{P}$ of m=0 and helicity j depends on a field of transformations of the restricted Lorentz group $p \to \Lambda(A_p)$ ($A_p \in SL(2, \mathbb{C})$) such that, for every $p \in \Omega$, we have

$$\Lambda(A_p)k_0 = p \ . \tag{2.40}$$

Given the field $p \to \Lambda(A_p)$ the unitary irreducible representation of $\mathcal{P}$ of m=0 and helicity j, denoted by $U^{[j]}(A, a)$, is then

$$(U^{[j]}(A, a)G)(p) = \mathrm{e}^{ia.p} L^j(A_p^{-1} A A_{\Lambda(A)^{-1}p}) G(\Lambda(A)^{-1}p) \ . \tag{2.41}$$

where $G(.) \in L^2(\Omega, \frac{\mathrm{d}^3\mathbf{p}}{2|\mathbf{p}|})$. Recall that $p = (|\mathbf{p}|, \mathbf{p})$.

Two important choices of A_p are made in physics.

2.2.1. *The canonical formalism.* This formalism corresponds to the choice made by A.S.Wightman (see [53] and [29]):

$$A_p^1 = \begin{pmatrix} \sqrt{\frac{|\mathbf{p}| + p^3}{2}} & 0 \\ \frac{p^1 + ip^2}{\sqrt{2(|\mathbf{p}| + p^3)}} & \sqrt{\frac{2}{|\mathbf{p}| + p^3}} \end{pmatrix} \tag{2.42}$$

2.2.2. *The helicity formalism.* In that case A_p^2 is the 2×2 matrix in $SL(2, \mathbb{C})$ corresponding to the product of a pure Lorentz transformation $\Lambda^H(A_p^2)$ such that

$$\Lambda^H(A_p^2)k_0 = (|\mathbf{p}|, 0, 0, |\mathbf{p}|) \ . \tag{2.43}$$

and of the same rotation R^h as the one defined for a positive mass. We then obtain

$$A_p^2 = \begin{pmatrix} |\mathbf{p}|^{\frac{1}{2}}\mathrm{e}^{-i\frac{\varphi}{2}} \cos\frac{\theta}{2} & -|\mathbf{p}|^{-\frac{1}{2}}\mathrm{e}^{-i\frac{\varphi}{2}} \sin\frac{\theta}{2} \\ |\mathbf{p}|^{\frac{1}{2}}\mathrm{e}^{i\frac{\varphi}{2}} \sin\frac{\theta}{2} & |\mathbf{p}|^{-\frac{1}{2}}\mathrm{e}^{i\frac{\varphi}{2}} \cos\frac{\theta}{2} \end{pmatrix} \tag{2.44}$$

θ(resp.φ) is the polar(resp.azimutal) angle of $\mathbf{p}$ with $0 \leq \theta \leq \pi$ (resp.$0 \leq \varphi < 2\pi$). See [47], [48], [29], [30] and [28] .

Remark 2.2. *The helicity j is Lorentz invariant. Nevertheless note that photons and gravitons have helicity ± 1 and ± 2 respectfully because of the symmetry of space inversion of the electromagnetic and gravitational interactions. On the other hand it is well known that the weak interactions do not respect the symmetry of space inversion. Thus one has to distinguish the neutrinos with helicity $-\frac{1}{2}$ from the antineutrinos with helicity $\frac{1}{2}$ in the Standard Model. It is conventional to call a particle with helicity $j > 0$ right-handed and a particle with helicity $-j$ left-handed.*

2.3. **The representations of the Poincaré group in $L^2(\mathbb{R}^3, \mathbb{C}^{2j+1})$ and in $L^2(\mathbb{R}^3)$.**

For most applications to Quantum Field Theory it is more convenient to use the spaces $L^2(\mathbb{R}^3, \mathbb{C}^{2j+1})$ for $m > 0$ and $L^2(\mathbb{R}^3)$ for $m = 0$ instead of the spaces $L^2(\Omega_m, \frac{\mathrm{d}^3\mathbf{p}}{2\omega_{\mathbf{p}}}, \mathbb{C}^{2j+1})$ and $L^2(\Omega, \frac{\mathrm{d}^3\mathbf{p}}{2|\mathbf{p}|})$ respectfully.

The following map

$$(V_m f)(\mathbf{p}) = (2\omega_{\mathbf{p}})^{-\frac{1}{2}} f(\omega_{\mathbf{p}}, \mathbf{p}) \ . \tag{2.45}$$

is a unitary map from $L^2(\Omega_m, \frac{\mathrm{d}^3\mathbf{p}}{2\omega_{\mathbf{p}}}, \mathbb{C}^{2j+1})$ onto $L^2(\mathbb{R}^3, \mathbb{C}^{2j+1})$ and

$$(V_0 f)(\mathbf{p}) = (2\omega_{\mathbf{p}})^{-\frac{1}{2}} f(\omega_{\mathbf{p}}, \mathbf{p}) \ . \tag{2.46}$$

is a unitary map from $L^2(\Omega, \frac{\mathrm{d}^3\mathbf{p}}{2|\mathbf{p}|})$ onto $L^2(\mathbb{R}^3)$.

We have for both cases

$$(V_{\cdot}^{-1} g)(p^0, \mathbf{p}) = \sqrt{2p^0} g(\mathbf{p}) \ . \tag{2.47}$$

where $g(\mathbf{p}) \in L^2(\mathbb{R}^3, \mathbb{C}^{2j+1})$ when $m > 0$ with $p^0 = \omega_{\mathbf{p}}$ and where $g(\mathbf{p}) \in L^2(\mathbb{R}^3)$ when $m = 0$ with $p^0 = |\mathbf{p}|$.

For any field $p \to \Lambda(A_p)$ of Lorentz transformations such that, for $m > 0$ and $p \in \Omega_m$,

$$\Lambda(A_p) k_m = p \tag{2.48}$$

one easily gets the form of the unitary irreducible representation of $\mathcal{P}$ corresponding the mass $m > 0$ and spin j in the space $L^2(\mathbb{R}^3, \mathbb{C}^{2j+1})$.

We obtain, for $f(\mathbf{p})$ belonging to $L^2(\mathbb{R}^3, \mathbb{C}^{2j+1})$,

$$\begin{aligned} &\{(V_m U^{[m,j]}(A,a) V_m^{-1}) f\}(\mathbf{p}) \\ &= \Big(\frac{\omega_{\mathbf{p}_{\Lambda(A)^{-1}p}}}{\omega_{\mathbf{p}}}\Big)^{\frac{1}{2}} \mathrm{e}^{ia.p} D^j(A_p^{-1} A A_{\Lambda(A)^{-1}p}) f(\mathbf{p}_{\Lambda(A)^{-1}p}) \ . \end{aligned} \tag{2.49}$$

Here $p^0 = \omega_{\mathbf{p}}$, i.e., $p = (\omega_{\mathbf{p}}, \mathbf{p})$ and $\mathbf{p}_{\Lambda(A)^{-1}p}$ is the three-vector part of $\Lambda(A)^{-1}p$ such that

$$\Lambda(A)^{-1}p = (\omega_{\mathbf{p}_{\Lambda(A)^{-1}p}}\ \mathbf{p}_{\Lambda(A)^{-1}p}) \tag{2.50}$$

For any field of Lorentz transformations $p \to \Lambda(A_p)$ such that

$$\Lambda(A_p) k_0 = p \ , p \in \Omega. \tag{2.51}$$

we easily get in a similar way the unitary irreducible representations of $\mathcal{P}$ in the massless case for helicity j in the space $L^2(\mathbb{R}^3)$.

Thus we obtain, for $g(\mathbf{p})$ belonging to $L^2(\mathbb{R}^3)$,

$$\begin{aligned} &\{(V_0 U^{[j]}(A,a) V_0^{-1}) g\}(\mathbf{p}) \\ &= \Big(\frac{|\mathbf{p}_{\Lambda(A)^{-1}p}|}{|\mathbf{p}|}\Big)^{\frac{1}{2}} \mathrm{e}^{ia.p} L^j(A_p^{-1} A A_{\Lambda(A)^{-1}p}) g(\mathbf{p}_{\Lambda(A)^{-1}p}) \ . \end{aligned} \tag{2.52}$$

where $p = (|\mathbf{p}|, \mathbf{p})$ and $\Lambda(A)^{-1}p = (|\mathbf{p}_{\Lambda(A)^{-1}p}|, \mathbf{p}_{\Lambda(A)^{-1}p})$.

We now set

$$\widetilde{U}^{[m,j]}(A,a) = V_m U^{[m,j]}(A,a) V_m^{-1} \tag{2.53}$$

$$\widetilde{U}^{[j]}(A,a) = V_0 U^{[j]}(A,a) V_0^{-1}\ . \tag{2.54}$$

Remark 2.3. *In* [47], [48], [49], [50] *and* [51] *the irreducible representations of $\mathcal{P}$ are written down in the space of generalized eigenfunctions of momenta, spins and helicities denoted by $\Psi_{p,\sigma}$ and $\Psi_{p,j}$ respectively. From the mathematical point of view the corresponding space is a subspace of the space of distributions $\mathfrak{D}'(\mathbb{R}^3, \mathbb{C}^{2j+1})$ for $m > 0$ and spin j and of $\mathfrak{D}'(\mathbb{R}^3)$ for $m = 0$ and helicity j. From the knowledge of the representations $\widetilde{U}^{[m,j]}$ and $\widetilde{U}^{[j]}$ in the spaces $L^2(\mathbb{R}^3, \mathbb{C}^{2j+1})$ and $L^2(\mathbb{R}^3)$ respectively it is not difficult to get the corresponding representations in the spaces of distributions by duality. For simplicity we keep the same notations $\widetilde{U}^{[m,j]}$ and $\widetilde{U}^{[j]}$ for the representations in the spaces of distributions.*

In the massive case we get

$$\begin{aligned}&(\widetilde{U}^{[m,j]}(A,a))\Psi_{p,\sigma} = \\ &\Big(\frac{\omega_{\boldsymbol{p}_{\Lambda(A)p}}}{\omega_{\boldsymbol{p}}}\Big)^{\frac{1}{2}} \mathrm{e}^{ia.\Lambda(A)p} \sum_{-j\le\sigma'\le j} D^j_{\sigma'\sigma}(A^{-1}_{\Lambda(A)p} A A_p)\Psi_{\Lambda(A)p,\sigma'}\ .\end{aligned} \tag{2.55}$$

In the massless case, we obtain

$$\begin{aligned}&(\widetilde{U}^{[j]}(A,a))\Psi_{p,j} = \\ &\Big(\frac{|\boldsymbol{p}_{\Lambda(A)p}|}{|\boldsymbol{p}|}\Big)^{\frac{1}{2}} \mathrm{e}^{ia.\Lambda(A)p} L^j(A^{-1}_{\Lambda(A)p} A A_p)\Psi_{\Lambda(A)p,j}\ .\end{aligned} \tag{2.56}$$

(2.55) *and* (2.56) *are the representations of $\mathcal{P}$ given in* [51, 2.5.23 and 2.5.42].

Remark 2.4. *Let $\mathbf{P} = (\mathrm{P}^1, \mathrm{P}^2, \mathrm{P}^3)$ be the components of the momentum operators and let $\mathbf{J} = (\mathrm{J}^1, \mathrm{J}^2, \mathrm{J}^3)$ be the components of the angular momentum vector. Let us consider the massive case for a given spin j. In the canonical formalism we obtain a spectral representation of the maximal set of commuting self-adjoint operators generated by $(\mathbf{P}, \mathrm{S}^3)$ where S^3 is J^3 in the rest frame of the particle generated by $\Lambda((A^c_p)^{-1})$. In the helicity formalism one gets a spectral representation of the maximal set of commuting self-adjoint operators generated by $(\mathbf{P}, \mathrm{H}^3)$ where H^3 is the helicity operator $\big(\sqrt{\sum_{i=1}^3 (\mathrm{P}^i)^2}\big)^{-1}(\sum_{l=1}^3 \mathrm{P}^l \mathrm{J}^l)$. S^3 and H^3 have the same spectrum $(-j, -j+1, \ldots, j-1, j)$.*

3. Free causal fields for a massive particle of any spin

In this chapter we now introduce the construction of unique free causal fields for particles with $m > 0$ and spin j. For that we follow the formalism of S.Weinberg as described in [47], [48], [49], [50] and [51, chapter 5]. See also [42].

3.1. **Fock spaces for massive particles of any spin.**

This is the most important case.Consider a particle with mass $m > 0$ and spin j.

Set

$$\mathbb{Z}_j = (-j, -j+1, \ldots, j-1, j) \tag{3.1}$$

and

$$\Sigma_j = \mathbb{R}^3 \times \mathbb{Z}_j \tag{3.2}$$

In the following $(\mathbf{p}, s)$ will be the quantum variables for a massive particle of spin j and for both the canonical and helicity formalisms. Here $\mathbf{p} \in \mathbb{R}^3$ and $s \in \mathbb{Z}_j$.In the sequel,we will identify $L^2(\mathbb{R}^3, \mathbb{C}^{2j+1})$ with $L^2(\Sigma_j)$. For simplicity we keep the same notations $\widetilde{U}^{[m,j]}$ for the representations of $\mathcal{P}$ in these two Hilbert spaces. We shall sometimes use the notations $\xi = (\mathbf{p}, s)$ and $\int_{\Sigma_j} \mathrm{d}\xi = \sum_{s\in\Sigma_j} \int \mathrm{d}^3\mathbf{p}$.

Let $\mathfrak{F}_s^{[m,j]}$ (resp.$\mathfrak{F}_a^{[m,j]}$) be the bosonic (resp. fermionic)Fock space for bosons(resp.fermions) of mass $m > 0$ and spin j. We have

$$\mathfrak{F}_s^{[m,j]} = \bigoplus_{n=0}^{\infty} \otimes_s^n L^2(\Sigma_j)\ . \tag{3.3}$$

where $\otimes_s^n$ denotes the symmetric n-th tensor product and $\otimes_s^0 L^2(\Sigma_j)$ $=\mathbb{C}$, and

$$\mathfrak{F}_a^{[m,j]} = \bigoplus_{n=0}^{\infty} \otimes_a^n L^2(\Sigma_j)\ . \tag{3.4}$$

where $\otimes_a^n$ denotes the antisymmetric n-th tensor product and $\otimes_a^0 L^2(\Sigma_j)$ $=\mathbb{C}$.

In the case where a massive particle has an antiparticle we introduce a Fock space for both the particles and antiparticles denoted by $\widetilde{\mathfrak{F}}_s^{[m,j]}$ and $\widetilde{\mathfrak{F}}_a^{[m,j]}$ respectively and defined by

$$\begin{aligned} \widetilde{\mathfrak{F}}_s^{[m,j]} &= \mathfrak{F}_s^{[m,j]} \otimes \mathfrak{F}_s^{[m,j]}\ , \\ \widetilde{\mathfrak{F}}_a^{[m,j]} &= \mathfrak{F}_s^{[m,j]} \otimes \mathfrak{F}_a^{[m,j]}\ . \end{aligned} \tag{3.5}$$

In the case of N bosons and N fermions with masses $(m_i)_{1\le i\le N}$ and spins $(j_i)_{1\le i\le N}$ the corresponding Fock spaces, denoted by $\mathfrak{F}_s^{(N)}$ and $\mathfrak{F}_a^{(N)}$ respectively, are given by

$$\mathfrak{F}_s^{(N)} = \bigotimes_{i=1}^{N} \mathfrak{F}_s^{[m_i,j_i]} \tag{3.6}$$

and

$$\mathfrak{F}_a^{(N)} = \bigotimes_{i=1}^{N} \mathfrak{F}_a^{[m_i,j_i]} \tag{3.7}$$

When particles and antiparticles are involved the corresponding Fock spaces are denoted $\widetilde{\mathfrak{F}}_s^{(N)}$ and $\widetilde{\mathfrak{F}}_a^{(N)}$ respectively and defined by

$$\begin{aligned} \widetilde{\mathfrak{F}}_s^{(N)} &= \bigotimes_{i=1}^{N} \widetilde{\mathfrak{F}}_s^{[m_i,j_i]}\ , \\ \widetilde{\mathfrak{F}}_a^{(N)} &= \bigotimes_{i=1}^{N} \widetilde{\mathfrak{F}}_a^{[m_i,j_i]}\ . \end{aligned} \tag{3.8}$$

The unitary irreducible representations $\widetilde{U}^{[m,j]}$ of $\mathcal{P}$ induce two unitary representations of $\mathcal{P}$ in $\mathfrak{F}_s^{[m,j]}$ and $\mathfrak{F}_a^{[m,j]}$ which are denoted by $\Gamma(\widetilde{U}^{[m,j]})$ where $\Gamma(.)$ is defined, for example, in [40, section X.7], [18, 4.53] and [16, 5.48].The unitary representation of $\mathcal{P}$ in $\widetilde{\mathfrak{F}}_s^{[m,j]}$ and $\widetilde{\mathfrak{F}}_a^{[m,j]}$ respectively is then $\Gamma(\widetilde{U}^{[m,j]} \otimes \widetilde{U}^{[m,j]})$.

We now introduce the creation and annihilation operators for bosons and fermions.

$a_\epsilon(\xi; m, j)$ (resp. $a^*_\epsilon(\xi; m, j)$) is the annihilation (resp.creation) operator for a massive boson of mass $m > 0$ and spin j if $\epsilon = +$ and for the corresponding massive antiparticle if $\epsilon = -$.

In the case where a particle is its own antiparticle $a(\xi; m, j)$ (resp. $a^*(\xi; m, j)$) is the annihilation (resp. creation) operator for the corresponding particle.

Similarly, $b_\epsilon(\xi; m, j)$ (resp. $b^*_\epsilon(\xi; m, j)$) is the annihilation (resp.creation) operator for a massive fermion of mass $m > 0$ and spin j if $\epsilon = +$ and for the corresponding massive antiparticle if $\epsilon = -$.

In the case where a particle is its own antiparticle $b(\xi; m, j)$ (resp. $b^*(\xi; m, j)$) is the annihilation (resp. creation) operator for the corresponding particle.

See [40, section X.7], [18, section 4.5], [9] and [16, section 5.4] for the definition of annihilation and creation operators.

The operators $a_\epsilon(\xi; m, j)$ and $a^*_\epsilon(\xi; m, j)$ fulfil the usual commutation relations (CCR), whereas $b_\epsilon(\xi; m, j)$ and $b^*_\epsilon(\xi; m, j)$ fulfil the canonical anticommutation relation (CAR). See [51]. Futhermore, the $a's$ commute with the $b's$.

In addition, in the case where several fermions are involved, we follow the convention described in [51, sections 4.1 and 4.2]. This means that we will assume that fermionic annihilation and creation operators of different species of particles anticommute (see [9, arXiv] for explicit definitions).

Therefore, the following canonical anticommutation and commutation relations hold for a couple of massive particles with $m > 0$ and $m' > 0$ and spins j and j',

$$\begin{aligned}
&\{b_\epsilon(\xi; m, j), b^*_{\epsilon'}(\xi'; m', j')\} = \delta_{\epsilon\epsilon'}\delta_{jj'}\delta_{mm'}\delta(\xi - \xi')\ ,\\
&[a_\epsilon(\xi; m, j), a^*_{\epsilon'}(\xi'; m', j')] = \delta_{\epsilon\epsilon'}\delta_{jj'}\delta_{mm'}\delta(\xi - \xi')\ ,\\
&\{b_\epsilon(\xi; m, j), b_{\epsilon'}(\xi'; m', j')\} = 0\ ,\\
&[a_\epsilon(\xi; m, j), a_{\epsilon'}(\xi'; m', j')] = 0\ ,\\
&[b_\epsilon(\xi; m, j), a_{\epsilon'}(\xi'; m', j')] = [b_\epsilon(\xi; m, j), a^*_{\epsilon'}(\xi'; m', j')] = 0\ .
\end{aligned} \tag{3.9}$$

where $\{b, b'\} = bb' + b'b$ and $[a, a'] = aa' - a'a$.

We now introduce

$$\begin{aligned}
a_\epsilon(m, j)(\varphi) &= \int_{\Sigma_j} a_\epsilon(\xi; m, j)\overline{\varphi(\xi)}\mathrm{d}\xi\ , a^*_\epsilon(m, j)(\varphi) = \int_{\Sigma_j} a^*_\epsilon(\xi; m, j)\varphi(\xi)\mathrm{d}\xi\ ,\\
b_\epsilon(m, j)(\varphi) &= \int_{\Sigma_j} b_\epsilon(\xi; m, j)\overline{\varphi(\xi)}\mathrm{d}\xi\ , b^*_\epsilon(m, j)(\varphi) = \int_{\Sigma_j} b^*_\epsilon(\xi; m, j)\varphi(\xi)\mathrm{d}\xi\ .
\end{aligned} \tag{3.10}$$

We recall that, for $\varphi \in L^2(\Sigma_j)$, the operators $b_\epsilon(m, j)$ and $b^*_\epsilon(m, j)$ are bounded operators on $\mathfrak{F}_a^{[m,j]}$ satisfying

$$\|b^\sharp_\epsilon(m, j)(\varphi)\| = \|\varphi\|_{L^2}\ . \tag{3.11}$$

where $b^\sharp$ is b or b^*.

We now study the transformation rules of the annihilation and creation operators by $\Gamma(\widetilde{U}^{[m,j]})$.

By[15, Lemma 2.7](see also [14, thm 18] and [18, 4.54]) we obtain for $f \in L^2(\Sigma_j)$

$$\Gamma(\widetilde{U}^{[m,j]}(A, a))a^*_\epsilon(m, j)(f)\Gamma(\widetilde{U}^{[m,j]}(A, a))^{-1} = a^*_\epsilon(m, j)(\widetilde{U}^{[m,j]}(A, a)f) \tag{3.12}$$

We now use the explicit notation $(\mathbf{p}, s)$ for ξ.

Note that, for $B \in SU(2)$,

$$D^j(B) = (D^j(B^{-1}))^* \tag{3.13}$$

where T^* is the adjoint of the operator T.

By (3.10) and (3.12) we get

$$\begin{aligned}
&\Gamma(\widetilde{U}^{[m,j]}(A,a))a^*_\epsilon(m,j)(f)\Gamma(\widetilde{U}^{[m,j]}(A,a))^{-1}\\
&= \sum_s \int \Big(\Gamma(\widetilde{U}^{[m,j]}(A,a))a^*_\epsilon(\mathbf{p},s;m,j)\Gamma(\widetilde{U}^{[m,j]}(A,a))^{-1}\Big) f_s(\mathbf{p})\mathrm{d}^3\mathbf{p}\\
&= \sum_s \int a^*_\epsilon(\mathbf{p},s;m,j)\Big(\widetilde{U}^{[m,j]}(A,a)f\Big)_s(\mathbf{p})\mathrm{d}^3\mathbf{p}\ .
\end{aligned} \tag{3.14}$$

By (2.49), (2.50), (3.13) and (3.14) we easily obtain

$$\begin{aligned}
&\Gamma(\widetilde{U}^{[m,j]}(A,a))a^*_\epsilon(\mathbf{p},s;m,j)\Gamma(\widetilde{U}^{[m,j]}(A,a))^{-1}\\
&= \sum_{s'} \Big(\frac{\omega_{\mathbf{p}_{\Lambda(A)p}}}{\omega_{\mathbf{p}}}\Big)^{\frac12} \mathrm{e}^{ia.\Lambda(A)p}\overline{D^j_{ss'}(A_p^{-1}AA_{\Lambda(A)p})}a^*_\epsilon(\mathbf{p}_{\Lambda(A)p},s';m,j)\ .
\end{aligned} \tag{3.15}$$

$\overline{z}$ is the complex conjugate of any number z .

By taking the adjoint of (3.15) we get

$$\begin{aligned}
&\Gamma(\widetilde{U}^{[m,j]}(A,a))a_\epsilon(\mathbf{p},s;m,j)\Gamma(\widetilde{U}^{[m,j]}(A,a))^{-1}\\
&= \sum_{s'} \Big(\frac{\omega_{\mathbf{p}_{\Lambda(A)p}}}{\omega_{\mathbf{p}}}\Big)^{\frac12} \mathrm{e}^{-ia.\Lambda(A)p}D^j_{ss'}(A_p^{-1}AA_{\Lambda(A)p})a_\epsilon(\mathbf{p}_{\Lambda(A)p},s';m,j)\ .
\end{aligned} \tag{3.16}$$

(3.16) and (3.15) are the equations (5.1.11) and (5.1.12) given in [51] written down with our choice of the space-time metric (2.2) instead of the one used by S.Weinberg in [51].

By [14, thm 18] we also have

$$\begin{aligned}
&\Gamma(\widetilde{U}^{[m,j]}(A,a))b^*_\epsilon(\mathbf{p},s;m,j)\Gamma(\widetilde{U}^{[m,j]}(A,a))^{-1}\\
&= \sum_{s'} \Big(\frac{\omega_{\mathbf{p}_{\Lambda(A)p}}}{\omega_{\mathbf{p}}}\Big)^{\frac12} \mathrm{e}^{ia.\Lambda(A)p}\overline{D^j_{ss'}(A_p^{-1}AA_{\Lambda(A)p})}b^*_\epsilon(\mathbf{p}_{\Lambda(A)p},s';m,j)\ .
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&\Gamma(\widetilde{U}^{[m,j]}(A,a))b_\epsilon(\mathbf{p},s;m,j)\Gamma(\widetilde{U}^{[m,j]}(A,a))^{-1}\\
&= \sum_{s'} \Big(\frac{\omega_{\mathbf{p}_{\Lambda(A)p}}}{\omega_{\mathbf{p}}}\Big)^{\frac12} \mathrm{e}^{-ia.\Lambda(A)p}D^j_{ss'}(A_p^{-1}AA_{\Lambda(A)p})b_\epsilon(\mathbf{p}_{\Lambda(A)p},s';m,j)\ .
\end{aligned} \tag{3.18}$$

Note that, in (3.12),(3.14),(3.15), (3.16), (3.17) and (3.18), A_p is A_p^C or A_p^H depending on the formalism we consider. It is important to remark that the operators of creation and annihilation both in the canonical and helicity formalism depend on the formalism we consider.

We further note that

$$^C\widetilde{U}^{[m,j]}(A,a) = D^j(A^{C-1}_{\cdot}A^H_{\cdot})\,{}^H\widetilde{U}^{[m,j]}(A,a)D^j(A^{C-1}_{\cdot}A^H_{\cdot})^{-1} \tag{3.19}$$

In view of (2.17) we get

$$A(\chi,\frac{\mathbf{p}}{|\mathbf{p}|}) = A(\mathbf{k}\wedge\frac{\mathbf{p}}{|\mathbf{p}|},\theta)A(\chi,\mathbf{k})A(\mathbf{k}\wedge\frac{\mathbf{p}}{|\mathbf{p}|},\theta)^{-1} \tag{3.20}$$

In view of (2.27) we set

$$B_{\frac{\mathbf{p}}{|\mathbf{p}|}} = \begin{pmatrix} \mathrm{e}^{-i\frac{\varphi}{2}}\cos\frac{\theta}{2} & -\mathrm{e}^{-i\frac{\varphi}{2}}\sin\frac{\theta}{2}\\ \mathrm{e}^{i\frac{\varphi}{2}}\sin\frac{\theta}{2} & \mathrm{e}^{i\frac{\varphi}{2}}\cos\frac{\theta}{2}\end{pmatrix} \tag{3.21}$$

Combining (3.20) with (3.19) we obtain

$$\begin{aligned} a_\epsilon^{C*}(\mathbf{p},s;m,j) &= \sum_{s'} D^j_{s's}\big((B_{\frac{\mathbf{p}}{|\mathbf{p}|}})^{-1}\big)a_\epsilon^{H*}(\mathbf{p},s';m,j) \\ a_\epsilon^{H*}(\mathbf{p},s;m,j) &= \sum_{s'} D^j_{s's}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})a_\epsilon^{C*}(\mathbf{p},s';m,j) \\ a_\epsilon^{C}(\mathbf{p},s;m,j) &= \sum_{s'} D^j_{s's}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})a_\epsilon^{H}(\mathbf{p},s';m,j) \\ a_\epsilon^{H}(\mathbf{p},s;m,j) &= \sum_{s'} D^j_{s's}\big((B_{\frac{\mathbf{p}}{|\mathbf{p}|}})^{-1}\big)a_\epsilon^{C}(\mathbf{p},s';m,j)\ . \end{aligned} \tag{3.22}$$

and likewise for $b_\epsilon^*(\mathbf{p},s;m,j)$ and $b_\epsilon(\mathbf{p},s;m,j)$.

In the following we will omit the superscripts C and H for $a^\sharp$ and $b^\sharp$ when the formalism that we are using is well determined.

The construction of free causal fields associated with a massive particle of spin j depends on the knowledge of the irreducible finite dimensional representations of $SL(2,\mathbb{C})$ that we now study.

3.2. **The irreducible finite dimensional representations of $SL(2,\mathbb{C})$.** These representations are well known. See, for example, [36] and [44]. Once again we shall follow the method used by S.Weinberg (see [51, subsection 5.6]) in order to construct such representations.

Let us recall the Lie algebra of $SL(2,\mathbb{C})$.

Let $\widetilde{\mathrm{J}}_i$, $i=1,2,3$, be the generators of the rotations and let $\widetilde{\mathrm{K}}_i$, $i=1,2,3$, be the generators of the pure Lorentz transformations.

We have

$$\begin{aligned} [\widetilde{\mathrm{J}}_i,\widetilde{\mathrm{J}}_j] &= i\epsilon_{ijk}\widetilde{\mathrm{J}}_k\ . \\ [\widetilde{\mathrm{J}}_i,\widetilde{\mathrm{K}}_j] &= i\epsilon_{ijk}\widetilde{\mathrm{K}}_k\ . \\ [\widetilde{\mathrm{K}}_i,\widetilde{\mathrm{K}}_j] &= -i\epsilon_{ijk}\widetilde{\mathrm{J}}_k\ . \end{aligned} \tag{3.23}$$

where ϵ_{ijk} is totally antisymmetric with $\epsilon_{123}=+1$.

The generators $\widetilde{\mathrm{J}}_i$ and $\widetilde{\mathrm{K}}_j$, $(i,j=1,2,3)$, satisfying (3.23) generate the Lie algebra of $\mathcal{L}$ and $SL(2,\mathbb{C})$.

In a given finite dimensional representation of the Lie algebra of $SL(2,\mathbb{C})$, $\mathrm{e}^{-i\theta(\sum_{l=1}^3 n^l \mathrm{J}_l)}$ is the representation of a lift in $SL(2,\mathbb{C})$ of the rotation of axis $\mathbf{n}=(n^1,n^2,n^3)$ and angle θ and $\mathrm{e}^{-i\chi(\sum_{l=1}^3 m^l \mathrm{K}_l)}$ is the representation of a lift in $SL(2,\mathbb{C})$ of the pure Lorentz transformation in the direction $\mathbf{m}=(m^1,m^2,m^3)$ and with rapidity $v=th\chi$. Here J_l and K_l, $l=1,2,3$, are the representations of $\widetilde{\mathrm{J}}_l$ and $\widetilde{\mathrm{K}}_l$.

We now introduce

$$\begin{aligned} \widetilde{\mathrm{M}}_{ij} &= -\widetilde{\mathrm{M}}_{ji} = \epsilon_{ijk}\widetilde{\mathrm{J}}_k \\ \widetilde{\mathrm{M}}_{i0} &= -\widetilde{\mathrm{M}}_{0i} = \widetilde{\mathrm{K}}_i \\ \widetilde{\mathrm{M}}_{00} &= \widetilde{\mathrm{M}}_{ii} = 0. \end{aligned} \tag{3.24}$$

Equations (3.23) and (3.24) then read

$$\big[\widetilde{\mathrm{M}}_{\mu\nu},\widetilde{\mathrm{M}}_{\rho\sigma}\big] = i\big(g_{\mu\sigma}\widetilde{\mathrm{M}}_{\nu\varrho} + g_{\nu\rho}\widetilde{\mathrm{M}}_{\mu\sigma} - g_{\nu\sigma}\widetilde{\mathrm{M}}_{\mu\rho} - g_{\mu\rho}\widetilde{\mathrm{M}}_{\nu\sigma}\big). \tag{3.25}$$

where μ,ν,ρ and σ run over the values 0,1,2,3.

Note that
$$\widetilde{\mathrm{M}}_{\mu\nu} = -\widetilde{\mathrm{M}}_{\nu\mu}. \tag{3.26}$$

The generators $\widetilde{\mathrm{M}}_{\mu\nu}$, $(\mu, \nu = 0, 1, 2, 3)$, satisfying (3.23) and (3.24) generate also the Lie algebra of $SL(2, \mathbb{C})$.

Any $A \in SL(2, \mathbb{C})$ can be written down in the following form
$$A = \mathrm{e}^{-\frac{i}{2}\omega^{\mu\nu}\mathrm{M}_{\mu\nu}} \tag{3.27}$$
where
$$\begin{aligned} &\omega^{\mu\nu} = -\omega^{\nu\mu} \\ &\mathrm{M}_{ij} = -\mathrm{M}_{ji} = \epsilon_{ijk}\frac{\sigma_k}{2} \\ &\mathrm{M}_{i0} = -\mathrm{M}_{0i} = i\frac{\sigma_i}{2}. \end{aligned} \tag{3.28}$$
In the case of a lift in $SL(2, \mathbb{C})$ of a rotation of axis $\mathbf{n}$ and angle θ, we have
$$\omega^{ij} = \epsilon^{ijk} n^k \theta \ , \omega^{i0} = 0. \tag{3.29}$$
and,in the case of a lift in $SL(2, \mathbb{C})$ of a pure Lorentz transformation in the direction $\mathbf{m}$ and with rapidity $v = \tanh\chi$, we have
$$\omega^{i0} = m^i \chi \ , \omega^{ij} = 0. \tag{3.30}$$
In view of (3.27) and (3.28) we have, for any $A \in SL(2, \mathbb{C})$,
$$\begin{aligned} &\mathrm{J}_i = \frac{\sigma_i}{2} \text{ and } \mathrm{K}_i = i\frac{\sigma_i}{2} \\ &A = \mathrm{e}^{\chi(\sum_{l=1}^{3} m^l \frac{\sigma_l}{2})}\mathrm{e}^{-i\theta(\sum_{l=1}^{3} n^l \frac{\sigma_l}{2})} \end{aligned} \tag{3.31}$$

We now introduce for $j = (1, 2, 3)$,
$$\begin{aligned} \mathcal{A}_j &= \frac{1}{2}(\widetilde{\mathrm{J}}_j + i\widetilde{\mathrm{K}}_j) \ . \\ \mathcal{B}_j &= \frac{1}{2}(\widetilde{\mathrm{J}}_j - i\widetilde{\mathrm{K}}_j) \ . \end{aligned} \tag{3.32}$$

We have
$$\begin{aligned} &[\mathcal{A}_i, \mathcal{A}_j] = i\epsilon_{ijk}\mathcal{A}_k \ . \\ &[\mathcal{B}_i, \mathcal{B}_j] = i\epsilon_{ijk}\mathcal{B}_k \ . \\ &[\mathcal{A}_i, \mathcal{B}_j] = 0 \ . \end{aligned} \tag{3.33}$$

By (3.33) the irreducible finite dimensional representations of $SL(2, \mathbb{C})$ are characterized by a couple of two positive integers and/or half-integers (J_1, J_2) representing the spins of two uncoupled particles.The generators of the spin J_1 are denoted by $\mathcal{J}_1^{(1)}$, $\mathcal{J}_2^{(1)}$, $\mathcal{J}_3^{(1)}$ and likewise for the spin J_2. The associated representation of $SL(2, \mathbb{C})$ will be denoted $D^{[J_1, J_2]}(.)$ where $D^{[J_1, J_2]}(A)$ is a matrix defined on $\mathbb{C}^{(2J_1+1)(2J_2+1)}$.

$\mathcal{J}_{.}^{(1)}$ are represented by the standard spin matrices for spin J^1. We have
$$\begin{aligned} &(\mathcal{J}_3^{(1)})_{M_1, M_1'} = M_1 \delta_{M_1, M_1'} \\ &(\mathcal{J}_1^{(1)} \pm i\mathcal{J}_2^{(1)})_{M_1, M_1'} = \delta_{M_1, M_1' \pm 1}\sqrt{J_1(J_1+1) - M_1'(M_1' \pm 1)} \ . \end{aligned} \tag{3.34}$$
where $M_1, M_1' \in (-J_1, -J_1+1, \ldots, J_1-1, J_1)$ and likewise for $\mathcal{J}_{.}^{(2)}$.

The matrices of $\mathcal{A}_.$ and $\mathcal{B}_.$ with respect to the tensor product of the canonical basis for the spins J_1 and J_2 are now given by

$$(\mathcal{A}_.)_{M_1M_2,M_1'M_2'} = \delta_{M_2,M_2'}(\mathcal{J}_.^{(1)})_{M_1,M_1'} \ , \qquad (\mathcal{B}_.)_{M_1M_2,M_1'M_2'} = \delta_{M_1,M_1'}(\mathcal{J}_.^{(2)})_{M_2,M_2'} \ . \tag{3.35}$$

We have

$$(\widetilde{\mathrm{J}}_.)_{M_1M_2,M_1'M_2'} = \delta_{M_2,M_2'}(\mathcal{J}_.^{(1)})_{M_1,M_1'} + \delta_{M_1,M_1'}(\mathcal{J}_.^{(2)})_{M_2,M_2'} \ . \qquad (\widetilde{\mathrm{K}}_.)_{M_1M_2,M_1'M_2'} = -i\delta_{M_2,M_2'}(\mathcal{J}_.^{(1)})_{M_1,M_1'} + i\delta_{M_1,M_1'}(\mathcal{J}_.^{(2)})_{M_2,M_2'} \ . \tag{3.36}$$

Note that, for $B \in SU(2)$, we have

$$D^{[J_1,J_2]}_{M_1M_2,M_1'M_2'}(B) = D^{J_1}_{M_1,M_1'}(B)D^{J_2}_{M_2,M_2'}(B) \ . \tag{3.37}$$

3.2.1. *Computation of $D^{[J_1,J_2]}(A^C_p)$.* It follows from (2.16),(2.23),(2.25),(3.27), (3.28),(3.30) and (3.31) that

$$D^{[J_1,J_2]}(A^C_p) = D^{[J_1,J_2]}(\mathrm{e}^{-i\chi\sum_{l=1}^3 \frac{p^l}{|\mathbf{p}|}\widetilde{\mathrm{K}}_l}) = D^{[J_1,J_2]}(\mathrm{e}^{-\chi\sum_{l=1}^3 \frac{p^l}{|\mathbf{p}|}(\mathcal{A}_l-\mathcal{B}_l)}) \ . \tag{3.38}$$

where, by (2.23), we have $\mathrm{e}^\chi = \frac{p^0}{m} + \frac{|\mathbf{p}|}{m} = \frac{|\mathbf{p}|+\omega_\mathbf{p}}{m}$.

By (3.35) and (3.38) we now get

$$D^{[J_1,J_2]}_{M_1M_2,M_1'M_2'}(A^C_p) = (\mathrm{e}^{-(\mathrm{Ln}\frac{|\mathbf{p}|+\omega_\mathbf{p}}{m})\sum_{l=1}^3 \frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(1)}})_{M_1M_1'}(\mathrm{e}^{(\mathrm{Ln}\frac{|\mathbf{p}|+\omega_\mathbf{p}}{m})\sum_{l=1}^3 \frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(2)}})_{M_2M_2'} \ . \tag{3.39}$$

3.2.2. *Computation of $D^{[J_1,J_2]}(A^H_p)$.* Recall that $B_{\frac{\mathbf{p}}{|\mathbf{p}|}}$, given by (3.21), is a lift in $SL(2,\mathbb{C})$ of the rotation of axis $\mathbf{k}\wedge\frac{\mathbf{p}}{|\mathbf{p}|}$ and angle θ=($\mathbf{k}$, $\frac{\mathbf{p}}{|\mathbf{p}|}$) where $\mathbf{k}$ is the unit vector of the third axis. By (2.14), (3.27), (3.28) and (3.29) we have

$$B_{\frac{\mathbf{p}}{|\mathbf{p}|}} = \mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\frac{\sigma_1}{2}+\frac{p^1}{|\mathbf{p}|}\frac{\sigma_2}{2})}. \tag{3.40}$$

According to the helicity formalism we have

$$A^H_p = B_{\frac{\mathbf{p}}{|\mathbf{p}|}}A^C_{(p^0,0,0,|\mathbf{p}|)} \tag{3.41}$$

By (3.34) and (3.39) we get

$$D^{[J_1,J_2]}_{M_1M_2M_1'M_2'}(A^C_{(p^0,0,0,|\mathbf{p}|)}) = (\frac{|\mathbf{p}|+\omega_\mathbf{p}}{m})^{M_2'-M_1'}\delta_{M_1M_1'}\delta_{M_2M_2'} \ . \tag{3.42}$$

By (3.37), (3.41) and (3.42) we finally get

$$D^{[J_1,J_2]}_{M_1M_2,M_1'M_2'}(A^H_p) = D^{J_1}_{M_1,M_1'}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})D^{J_2}_{M_2,M_2'}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})(\frac{|\mathbf{p}|+\omega_\mathbf{p}}{m})^{M_2'-M_1'} \ . \tag{3.43}$$

where

$$D^{J_l}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}}) = \mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(l)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(l)})} \ , l=1,2. \tag{3.44}$$

3.3. **Free causal fields for a massive particle of spin j.**

Consider a particle of mass $m > 0$ and spin j. Let (J_1, J_2) be two spins such that

$$|J_1 - J_2| \leq j \leq J_1 + J_2 \ . \tag{3.45}$$

One can prove the existence of unique causal free fields denoted by $\left({}^\sharp\Psi^{[J_1,J_2]\epsilon}_{M_1 M_2}(x)\right)_{M_1 M_2}$,where $M_1 \in (-J_1, -J_1+1, \dots, J_1-1, J_1)$ and $M_2 \in (-J_2, -J_2+1, \dots, J_2-1, J_2)$ and where $\sharp$=C or H and $\epsilon = \pm$, involving particles and antiparticles.

Set

$$\widetilde{V}^{[m,j]}(A,a) = \widetilde{U}^{[m,j]}(A,a) \oplus \widetilde{U}^{[m,j]}(A,a) \tag{3.46}$$

The causal free fields have to satisfy the two fundamental conditions:

(a)The relativistic covariance law:

$$\begin{aligned} &\big(\Gamma(\widetilde{V}^{[m,j]}(A,a))\big)\big({}^\sharp\Psi^{[J_1,J_2]\epsilon}_{M_1 M_2}\big)(x)\big(\Gamma(\widetilde{V}^{[m,j]}(A,a))\big)^{-1} \\ &= \sum_{M_1' M_2'} D^{[J_1,J_2]}_{M_1 M_2 M_1' M_2'}(A^{-1})\big({}^\sharp\Psi^{[J_1,J_2]\epsilon}_{M_1' M_2'}\big)(\Lambda(A)x + a) \ . \end{aligned} \tag{3.47}$$

where $x \in \mathbb{R}^4$.

(b)The microscopic causality in the bosonic case:

$$\begin{aligned} &[{}^\sharp\Psi^{[J_1,J_2]\epsilon}_{M_1 M_2}(x), {}^\sharp\Psi^{[J_1,J_2]\epsilon}_{M_1' M_2'}(y)] = \\ &= [{}^\sharp\Psi^{[J_1,J_2]\epsilon}_{M_1 M_2}(x), {}^\sharp\Psi^{[J_1,J_2]\epsilon\dagger}_{M_1' M_2'}(y)] = 0 \ , \end{aligned} \tag{3.48}$$

and

(c)The microscopic causality in the fermionic case:

$$\begin{aligned} &\{{}^\sharp\Psi^{[J_1,J_2]\epsilon}_{M_1 M_2}(x), {}^\sharp\Psi^{[J_1,J_2]\epsilon}_{M_1' M_2'}(y)\} = \\ &= \{{}^\sharp\Psi^{[J_1,J_2]\epsilon}_{M_1 M_2}(x), {}^\sharp\Psi^{[J_1,J_2]\epsilon\dagger}_{M_1' M_2'}(y)\} = 0 \ . \end{aligned} \tag{3.49}$$

for x-y space-like.

From now on we restrict ourselves to the case of a massive boson of spin j. We suppose that the massive boson is not its own antiparticle.The case of a massive fermion is strictly similar and we shall omit the details.Moreover when a particle is its own antiparticle the results are an easy consequence of what it follows.

Mimicking [51, chapter5] we set

$$\begin{aligned} &({}^\sharp_1\Upsilon^{[J_1,J_2]\epsilon}_{M_1 M_2})(x) \\ &= \sum_s \int \mathrm{d}^3\mathbf{p}({}^\sharp u^{[J_1,J_2]}_{M_1 M_2})(x;\mathbf{p},s;m,j)a_\epsilon(\mathbf{p},s;m,j) \ . \end{aligned} \tag{3.50}$$

and

$$\begin{aligned} &({}^\sharp_2\Upsilon^{[J_1,J_2]\epsilon'}_{M_1 M_2})(x) \\ &= \sum_s \int \mathrm{d}^3\mathbf{p}({}^\sharp v^{[J_1,J_2]}_{M_1 M_2})(x;\mathbf{p},s;m,j)a^*_{\epsilon'}(\mathbf{p},s;m,j) \ . \end{aligned} \tag{3.51}$$

Here $\epsilon \neq \epsilon'$.

$({}^\sharp_1\Upsilon^{[J_1,J_2]\epsilon}_{M_1 M_2})(x)$ and $({}^\sharp_2\Upsilon^{[J_1,J_2]\epsilon'}_{M_1 M_2})(x)$ are supposed to satisfy (3.47).

For simplicity we now omit the superscripts $[J_1,J_2]\epsilon$, $[J_1,J_2]\epsilon'$ and $[J_1,J_2]$. We will finally give the complete formulae later.

Combining (3.15) and (3.16) with (3.46), (3.47) and (3.50) we obtain

$$\begin{aligned}&\Big(\frac{p^0}{(\Lambda(A)p)^0}\Big)^{\frac{1}{2}}\sum_{M_1'M_2'}D_{M_1M_2M_1'M_2'}(A)(\mathrm{e}^{-ia.\Lambda(A)p})(^\sharp u_{M_1'M_2'})(x;\mathbf{p},s;m,j)\\&=\sum_{s'}D^j_{s's}(A^{\sharp-1}_{\Lambda(A)p}AA^\sharp_p)(^\sharp u_{M_1M_2})(\Lambda(A)x+a;\mathbf{p}_{\Lambda(A)p},s';m,j)\ .\end{aligned}\tag{3.52}$$

and

$$\begin{aligned}&\Big(\frac{p^0}{(\Lambda(A)p)^0}\Big)^{\frac{1}{2}}\sum_{M_1'M_2'}D_{M_1M_2M_1'M_2'}(A)(\mathrm{e}^{ia.\Lambda(A)p})(^\sharp v_{M_1'M_2'})(x;\mathbf{p},s;m,j)\\&=\sum_{s'}\overline{(D^j_{s's}(A^{\sharp-1}_{\Lambda(A)p}AA^\sharp_p))}(^\sharp v_{M_1M_2})(\Lambda(A)x+a;\mathbf{p}_{\Lambda(A)p},s';m,j)\ .\end{aligned}\tag{3.53}$$

By (3.52) and (3.53) with $A=1$ and for any $a\in\mathbb{R}^3$, $^\sharp u_{M_1M_2}(x;\mathbf{p},s;m,j)$ and $^\sharp v_{M_1M_2}(x;\mathbf{p},s;m,j)$ have the form $e^{-ia.x}(^\sharp u_{M_1M_2})(\mathbf{p},s;m,j)$ and $e^{ia.x}(^\sharp v_{M_1M_2})(\mathbf{p},s;m,j)$ respectively .

Following the convention in Physics we set (see [51, chapter5])

$$(^\sharp u_{M_1M_2})(x;\mathbf{p},s;m,j)=\big(2\pi\big)^{-3/2}e^{-ip.x}(^\sharp u_{M_1M_2})(\mathbf{p},s;m,j)\tag{3.54}$$

$$(^\sharp v_{M_1M_2})(x;\mathbf{p},s;m,j)=\big(2\pi\big)^{-3/2}e^{ip.x}(^\sharp v_{M_1M_2})(\mathbf{p},s;m,j)\ .\tag{3.55}$$

This, together with (3.52) and (3.53), yields

$$\begin{aligned}&\Big(\frac{p^0}{(\Lambda(A)p)^0}\Big)^{\frac{1}{2}}\sum_{M_1'M_2'}D_{M_1M_2M_1'M_2'}(A)(^\sharp u_{M_1'M_2'})(\mathbf{p},s;m,j)\\&=\sum_{s'}D^j_{s's}(A^{\sharp-1}_{\Lambda(A)p}AA^\sharp_p)(^\sharp u_{M_1M_2})(\mathbf{p}_{\Lambda(A)p},s';m,j)\ .\end{aligned}\tag{3.56}$$

and

$$\begin{aligned}&\Big(\frac{p^0}{(\Lambda(A)p)^0}\Big)^{\frac{1}{2}}\sum_{M_1'M_2'}D_{M_1M_2M_1'M_2'}(A)(^\sharp v_{M_1'M_2'})(\mathbf{p},s;m,j)\\&=\sum_{s'}\overline{(D^j_{s's}(A^{\sharp-1}_{\Lambda(A)p}AA^\sharp_p))}(^\sharp v_{M_1M_2})(\mathbf{p}_{\Lambda(A)p},s';m,j)\ .\end{aligned}\tag{3.57}$$

Letting $p=k_m$, where k_m is defined in (2.19), and $A=A^\sharp_p$ with $p\in\Omega_m$ in (3.56) and (3.57) one easily shows that

$$\begin{aligned}&(^\sharp u^{[J_1,J_2]}_{M_1M_2})(\mathbf{p},s;m,j)\\&=\Big(\frac{m}{p^0}\Big)^{\frac{1}{2}}\sum_{M_1'M_2'}D^{[J_1,J_2]}_{M_1M_2M_1'M_2'}(A^\sharp_p)(^\sharp u^{[J_1,J_2]}_{M_1'M_2'})(0,s;m,j)\ .\end{aligned}\tag{3.58}$$

and

$$
\begin{aligned}
&({}^{\sharp}v^{[J_1,J_2]}_{M_1M_2})(\mathbf{p},s;m,j)\\
&=\Big(\frac{m}{p^0}\Big)^{\frac{1}{2}}\sum_{M_1'M_2'}D^{[J_1,J_2]}_{M_1M_2M_1'M_2'}(A^{\sharp}_p)({}^{\sharp}v^{[J_1,J_2]}_{M_1'M_2'})(0,s;m,j)\ .
\end{aligned}
\tag{3.59}
$$

where we have introduced the superscript $[J_1,J_2]$ again and where $\sharp$=C or H .

By using (3.56) and (3.57) with $\mathbf{p}=0$ and $A\in SU(2)$ S.Weinberg shows that (see [51, section5.7])

$$
({}^{\sharp}u^{[J_1,J_2]}_{M_1M_2})(0,s;m,j)=\Big(\frac{1}{2m}\Big)^{\frac{1}{2}}(J_1J_2js|J_1M_1J_2M_2) \tag{3.60}
$$

$$
({}^{\sharp}v^{[J_1,J_2]}_{M_1M_2})(0,s;m,j)=(-1)^{j+s}({}^{\sharp}u^{[J_1,J_2]}_{M_1M_2})(0,-s;m,j)\ . \tag{3.61}
$$

where $(J_1J_2js|J_1M_1J_2M_2)$ is the Clebsch-Gordan coefficient in the notation of A.R.Edmonds (see [17]) .The Clebsch-Gordan coefficient vanishes unless $s=M_1+M_2$ so that we have

$$
(J_1J_2js|J_1M_1J_2M_2)=(J_1J_2js|J_1M_1J_2M_2)\delta_{s,M_1+M_2}. \tag{3.62}
$$

j is of the same type, integer or half-integer, as J_1+J_2 and $|J_1-J_2|$.

It follows from (3.39), (3.58), (3.59), (3.60) and (3.61) that, for the canonical formalism,

$$
\begin{aligned}
&({}^{C}u^{[J_1,J_2]}_{M_1M_2})(\mathbf{p},s;m,j)\\
&=\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_{M_1'M_2'}\big((\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^{3}\frac{p^l}{|\mathbf{p}|}\mathcal{J}^{(1)}_l})_{M_1M_1'}(\mathrm{e}^{\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^{3}\frac{p^l}{|\mathbf{p}|}\mathcal{J}^{(2)}_l})_{M_2M_2'}\\
&\times(J_1J_2js|J_1M_1'J_2M_2')\big).
\end{aligned}
\tag{3.63}
$$

and

$$
({}^{C}v^{[J_1,J_2]}_{M_1M_2})(\mathbf{p},s;m,j)=(-1)^{j+s}({}^{C}u^{[J_1,J_2]}_{M_1M_2})(\mathbf{p},-s;m,j)\ . \tag{3.64}
$$

By (3.43) we now get for the helicity formalism

$$
\begin{aligned}
&({}^{H}u^{[J_1,J_2]}_{M_1M_2})(\mathbf{p},s;m,j)\\
&=\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\sum_{M_1'M_2'}\big(D^{J_1}_{M_1M_1'}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})D^{J_2}_{M_2M_2'}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})\\
&\times(\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})^{M_2'-M_1'}(J_1J_2js|J_1M_1'J_2M_2')\big).
\end{aligned}
\tag{3.65}
$$

and

$$
({}^{H}v^{[J_1,J_2]}_{M_1M_2})(\mathbf{p},s;m,j)=(-1)^{j+s}({}^{H}u^{[J_1,J_2]}_{M_1M_2})(\mathbf{p},-s;m,j)\ . \tag{3.66}
$$

Together with (3.50), (3.51), (3.54), (3.55), (3.63), (3.64), (3.65) and (3.66) we now set

$$
{}^{\sharp}\Psi^{[J_1,J_2]\epsilon}_{M_1M_2}(.)=\alpha({}^{\sharp}_{1}\Upsilon^{[J_1,J_2]\epsilon}_{M_1M_2}(.))+\beta({}^{\sharp}_{2}\Upsilon^{[J_1,J_2]\epsilon}_{M_1M_2}(.)) \tag{3.67}
$$

${}^\sharp\Psi^{[J_1,J_2]\epsilon}_{M_1M_2}(x)$ satisfies the relativistic covariance law given by (3.47). In order to verify the microscopic causality condition given by (3.48) S.Weinberg has carefully shown that one must have $|\alpha|=|\beta|$ with

$$\beta=(-1)^{2J_2}\gamma\alpha \quad ,|\gamma|=1 \tag{3.68}$$

γ is the same for every field for a given particle.

α and γ can be eliminated so that we finally obtain in the bosonic case when $j\in\mathbb{N}$ and in the case of the canonical formalism

$$\begin{aligned}
&{}^C\Psi^{[J_1,J_2]\epsilon}_{M_1M_2}(x)\\
&=(2\pi)^{-\frac{3}{2}}\sum_s\int \mathrm{d}^3\mathbf{p}\Big(\frac{1}{\sqrt{2\omega_\mathbf{p}}}\Big(\sum_{M'_1M'_2}(\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_\mathbf{p}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}^{(1)}_l})_{M_1,M'_1}\\
&(\mathrm{e}^{(\ln\frac{|\mathbf{p}|+\omega_\mathbf{p}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}^{(2)}_l})_{M_2,M'_2}(J_1J_2js|J_1M'_1J_2M'_2)\Big)\times\\
&\mathrm{e}^{-ip.x}a_\epsilon(\mathbf{p},s;m,j)\\
&+(-1)^{2J_2+j+s}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\Big(\sum_{M'_1M'_2}(\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_\mathbf{p}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}^{(1)}_l})_{M_1,M'_1}\\
&(\mathrm{e}^{(\ln\frac{|\mathbf{p}|+\omega_\mathbf{p}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}^{(2)}_l})_{M_2,M'_2}\times(J_1J_2j(-s)|J_1M'_1J_2M'_2)\Big)\times\\
&\mathrm{e}^{ip.x}a^*_{\epsilon'}(\mathbf{p},s;m,j)\Big)\ .
\end{aligned}\tag{3.69}$$

We also have

$$\begin{aligned}
&{}^C\Psi^{[J_1,J_2]\epsilon}_{M_1M_2}(x)\\
&=(2\pi)^{-\frac{3}{2}}\sum_s\int \mathrm{d}^3\mathbf{p}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\Big(\sum_{M'_1M'_2}(\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_\mathbf{p}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}^{(1)}_l})_{M_1M'_1}\\
&\times(\mathrm{e}^{(\ln\frac{|\mathbf{p}|+\omega_\mathbf{p}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}^{(2)}_l})_{M_2M'_2}(J_1J_2js|J_1M'_1J_2M'_2)\Big)\\
&(\mathrm{e}^{-ip.x}a_\epsilon(\mathbf{p},s;m,j)+(-1)^{2J_2+j-s}\mathrm{e}^{ip.x}a^*_{\epsilon'}(\mathbf{p},-s;m,j)).
\end{aligned}\tag{3.70}$$

On the other hand we obtain in the case of the helicity formalism

$$\begin{aligned}
&{}^H\Psi^{[J_1,J_2]\epsilon}_{M_1M_2}(x)\\
&=(2\pi)^{-\frac{3}{2}}\sum_s\int \mathrm{d}^3\mathbf{p}\Big(\frac{1}{\sqrt{2\omega_\mathbf{p}}}\Big(\sum_{M'_1M'_2}D^{J_1}_{M_1,M'_1}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})D^{J_2}_{M_2,M'_2}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})\\
&(\frac{|\mathbf{p}|+\omega_\mathbf{p}}{m})^{M'_2-M'_1}(J_1J_2js|J_1M'_1J_2M'_2)\Big)\mathrm{e}^{-ip.x}a_\epsilon(\mathbf{p},s;m,j)\\
&+(-1)^{2J_2+j+s}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\Big(\sum_{M'_1M'_2}D^{J_1}_{M_1,M'_1}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})D^{J_2}_{M_2,M'_2}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})\\
&(\frac{|\mathbf{p}|+\omega_\mathbf{p}}{m})^{M'_2-M'_1}(J_1J_2j(-s)|J_1M'_1J_2M'_2)\Big)\mathrm{e}^{ip.x}a^*_{\epsilon'}(\mathbf{p},s;m,j)\Big)\ .
\end{aligned}\tag{3.71}$$

We also obtain

$$\begin{aligned}
&{}^{H}\Psi^{[J_1,J_2]\epsilon}_{M_1M_2}(x)\\
&= (2\pi)^{-\frac{3}{2}}\sum_s\int \mathrm{d}^3\mathbf{p}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\sum_{M_1'M_2'}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(1)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(1)})}\big)_{M_1M_1'}\\
&\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(2)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(2)})}\big)_{M_2M_2'}(\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})^{M_2'-M_1'}(J_1J_2js|J_1M_1'J_2M_2')\Big)\\
&(\mathrm{e}^{-ip.x}a_\epsilon(\mathbf{p},s;m,j)+(-1)^{2J_2+j-s}\mathrm{e}^{ip.x}a^*_{\epsilon'}(\mathbf{p},-s;m,j)).
\end{aligned} \tag{3.72}$$

Similarly, in the fermionic case when $j\in\mathbb{N}+1/2$, we obtain in the case of the canonical formalism

$$\begin{aligned}
&{}^{C}\widetilde{\Psi}^{[J_1,J_2]\epsilon}_{M_1M_2}(x)\\
&= (2\pi)^{-\frac{3}{2}}\sum_s\int \mathrm{d}^3\mathbf{p}\Bigg(\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\sum_{M_1'M_2'}(\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(1)}})_{M_1,M_1'}\\
&(\mathrm{e}^{(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(2)}})_{M_2,M_2'}(J_1J_2js|J_1M_1'J_2M_2')\Big)\mathrm{e}^{-ip.x}b_\epsilon(\mathbf{p},s;m,j)\\
&+(-1)^{2J_2+j+s}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\sum_{M_1'M_2'}(\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(1)}})_{M_1,M_1'}\\
&(\mathrm{e}^{(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(2)}})_{M_2,M_2'}(J_1J_2j(-s)|J_1M_1'J_2M_2')\Big)\mathrm{e}^{ip.x}b^*_{\epsilon'}(\mathbf{p},s;m,j)\Bigg)\ .
\end{aligned} \tag{3.73}$$

We also get

$$\begin{aligned}
&{}^{C}\widetilde{\Psi}^{[J_1,J_2]\epsilon}_{M_1M_2}(x)\\
&= (2\pi)^{-\frac{3}{2}}\sum_s\int \mathrm{d}^3\mathbf{p}\Bigg(\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\sum_{M_1'M_2'}(\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(1)}})_{M_1M_1'}\\
&\times(\mathrm{e}^{(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(2)}})_{M_2M_2'}(J_1J_2js|J_1M_1'J_2M_2')\Big)\\
&(\mathrm{e}^{-ip.x}b_\epsilon(\mathbf{p},s;m,j)+(-1)^{2J_2+j-s}\mathrm{e}^{ip.x}b^*_{\epsilon'}(\mathbf{p},-s;m,j))\Bigg).
\end{aligned} \tag{3.74}$$

and in the case of the helicity formalism

$$\begin{aligned}
&{}^{H}\widetilde{\Psi}^{[J_1,J_2]\epsilon}_{M_1M_2}(x) = (2\pi)^{-\frac{3}{2}}\sum_s\int \mathrm{d}^3\mathbf{p}\Bigg(\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\sum_{M_1'M_2'}D^{J_1}_{M_1,M_1'}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})D^{J_2}_{M_2,M_2'}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})\\
&\times(\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})^{M_2'-M_1'}(J_1J_2js|J_1M_1'J_2M_2')\Big)\mathrm{e}^{-ip.x}b_\epsilon(\mathbf{p},s;m,j)\\
&+(-1)^{2J_2+j+s}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\sum_{M_1'M_2'}D^{J_1}_{M_1,M_1'}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})D^{J_2}_{M_2,M_2'}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})\\
&\times(\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})^{M_2'-M_1'}(J_1J_2j(-s)|J_1M_1'J_2M_2')\Big)\mathrm{e}^{ip.x}b^*_{\epsilon'}(\mathbf{p},s;m,j)\Bigg)\ .
\end{aligned} \tag{3.75}$$

We also have

$$
\begin{aligned}
&{}^{H}\widetilde{\Psi}^{[J_1,J_2]\epsilon}_{M_1M_2}(x)\\
&= (2\pi)^{-\frac{3}{2}}\sum_{s}\int \mathrm{d}^3\mathbf{p}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\Big(\sum_{M_1'M_2'}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(1)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(1)})}\big)_{M_1M_1'}\\
&\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(2)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(2)})}\big)_{M_2M_2'}(\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})^{M_2'-M_1'}(J_1J_2js|J_1M_1'J_2M_2')\Big)\\
&(\mathrm{e}^{-ip.x}b_\epsilon(\mathbf{p},s;m,j)+(-1)^{2J_2+j-s}\mathrm{e}^{ip.x}b^*_{\epsilon'}(\mathbf{p},-s;m,j)).
\end{aligned}
\tag{3.76}
$$

Remark 3.1. *Note that the construction of the fields* ${}^{\sharp}\Psi^{[J_1,J_2]\epsilon}_{M_1M_2}(x)$ *involves an irreducible representation of* $SL(2,\mathbb{C})$ *of finite dimension.From a physical point of view, in particular in the case of an interaction invariant by space inversion, it can be more convenient to construct such fields associated to a direct sum of irreducible representations of finite dimension.For example the Dirac field for a particle of spin* $1/2$ *is based on the representation* $[\frac{1}{2},0]\oplus[0,\frac{1}{2}]$*.In particular the two cases* $[j,0]$ *and* $[0,j]$ *are important(see* [46]*).*

3.4. **Two particular cases: [j,0] and [0,j].**

In the bosonic case when $J_1=j\in\mathbb{N}$ and $J_2=0$ we have $(j0js|js'00)=\delta_{s,s'}$ for the Clebsch-Gordan coefficient and from (3.68) and (3.70) we obtain

$$
\begin{aligned}
&{}^{C}\Psi^{[j,0]\epsilon}_{s}(x) = (2\pi)^{-\frac{3}{2}}\sum_{s'}\int \mathrm{d}^3\mathbf{p}\Big(\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(j)}})_{ss'}\\
&\mathrm{e}^{-ip.x}a_\epsilon(\mathbf{p},s';m,j)+\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(j)}})_{s(-s')}\\
&(-1)^{j+s'}\mathrm{e}^{ip.x}a^*_{\epsilon'}(\mathbf{p},s';m,j))\Big)\ .
\end{aligned}
\tag{3.77}
$$

where $\big(\mathcal{J}_l^{(j)}\big)$, $l=1,2,3$, are the generators of the rotations in the representation $D^j(\mathbf{.})$ of $SU(2)$.

We also have

$$
\begin{aligned}
&{}^{C}\Psi^{[j,0]\epsilon}_{s}(x)\\
&= (2\pi)^{-\frac{3}{2}}\sum_{s'}\int \mathrm{d}^3\mathbf{p}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(j)}})_{ss'}\\
&\big(\mathrm{e}^{-ip.x}a_\epsilon(\mathbf{p},s';m,j)+(-1)^{j-s'}\mathrm{e}^{ip.x}a^*_{\epsilon'}(\mathbf{p},-s';m,j)\big)\ .
\end{aligned}
\tag{3.78}
$$

and

$$
\begin{aligned}
&{}^{H}\Psi^{[j,0]\epsilon}_{s}(x) = (2\pi)^{-\frac{3}{2}}\sum_{s'}\int \mathrm{d}^3\mathbf{p}\Big(\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(j)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(j)})}\big)_{ss'}\\
&(\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})^{-s'}\mathrm{e}^{-ip.x}a_\epsilon(\mathbf{p},s';m,j)+\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(j)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(j)})}\big)_{s(-s')}\\
&(\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})^{s'}(-1)^{j+s'}\mathrm{e}^{-ip.x}a^*_{\epsilon}(\mathbf{p},s';m,j)\Big)\ .
\end{aligned}
\tag{3.79}
$$

We also get

$$\begin{aligned}&{}^{H}\Psi_s^{[j,0]\epsilon}(x)\\&=(2\pi)^{-\frac{3}{2}}\sum_{s'}\int \mathrm{d}^3\mathbf{p}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(j)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(j)})}\big)_{ss'}\\&\big(\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m}\big)^{-s'}\big(\mathrm{e}^{-ip.x}a_\epsilon(\mathbf{p},s';m,j)+(-1)^{j-s'}\mathrm{e}^{ip.x}a_{\epsilon'}^*(\mathbf{p},-s';m,j)\big).\end{aligned}\tag{3.80}$$

In the fermionic case when $j\in\mathbb{N}+1/2$ we obtain

$$\begin{aligned}&{}^{C}\widetilde{\Psi}_s^{[j,0]\epsilon}(x)=(2\pi)^{-\frac{3}{2}}\sum_{s'}\int \mathrm{d}^3\mathbf{p}\Bigg(\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(j)}})_{ss'}\\&\mathrm{e}^{-ip.x}b_\epsilon(\mathbf{p},s';m,j)+\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(j)}})_{s(-s')}\\&(-1)^{j+s'}\mathrm{e}^{ip.x}b_{\epsilon'}^*(\mathbf{p},s';m,j))\Bigg)\ .\end{aligned}\tag{3.81}$$

We also get

$$\begin{aligned}&{}^{C}\widetilde{\Psi}_s^{[j,0]\epsilon}(x)\\&=(2\pi)^{-\frac{3}{2}}\sum_{s'}\int \mathrm{d}^3\mathbf{p}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(\mathrm{e}^{-(\ln\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m})\sum_{l=1}^3\frac{p^l}{|\mathbf{p}|}\mathcal{J}_l^{(j)}})_{ss'}\\&(\mathrm{e}^{-ip.x}b_\epsilon(\mathbf{p},s';m,j)+(-1)^{j-s'}\mathrm{e}^{ip.x}b_{\epsilon'}^*(\mathbf{p},-s';m,j)).\end{aligned}\tag{3.82}$$

and

$$\begin{aligned}&{}^{H}\widetilde{\Psi}_s^{[j,0]\epsilon}(x)=(2\pi)^{-\frac{3}{2}}\sum_{s'}\int \mathrm{d}^3\mathbf{p}\Bigg(\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(j)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(j)})}\big)_{ss'}\\&\big(\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m}\big)^{-s'}\mathrm{e}^{-ip.x}b_\epsilon(\mathbf{p},s';m,j)+\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(j)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(j)})}\big)_{s(-s')}\\&\big(\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m}\big)^{s'}(-1)^{j+s'}\mathrm{e}^{-ip.x}b_\epsilon^*(\mathbf{p},s';m,j)\Bigg)\ .\end{aligned}\tag{3.83}$$

Also

$$\begin{aligned}&{}^{H}\widetilde{\Psi}_s^{[j,0]\epsilon}(x)\\&=(2\pi)^{-\frac{3}{2}}\sum_{s'}\int \mathrm{d}^3\mathbf{p}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(j)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(j)})}\big)_{ss'}\\&\big(\frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m}\big)^{-s'}\big(\mathrm{e}^{-ip.x}b_\epsilon(\mathbf{p},s';m,j)+(-1)^{j-s'}\mathrm{e}^{ip.x}b_{\epsilon'}^*(\mathbf{p},-s';m,j)\big).\end{aligned}\tag{3.84}$$

In the bosonic case when $J_2=j\in\mathbb{N}$ and $J_1=0$ we have $(0jjs|00js')=\delta_{s,s'}$ for the Clebsch-Gordan coefficient and from (3.68) and (3.70) we obtain

$$
\begin{aligned}
&{}^{C}\Psi_s^{[0,j]\epsilon}(x) = (2\pi)^{-\frac{3}{2}} \sum_{s'} \int \mathrm{d}^3\mathbf{p} \Bigg(\frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (\mathrm{e}^{(\ln \frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m}) \sum_{l=1}^{3} \frac{p^l}{|\mathbf{p}|} \mathcal{J}_l^{(j)}})_{ss'} \\
&\mathrm{e}^{-ip.x} a_\epsilon(\mathbf{p}, s'; m, j) + \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (\mathrm{e}^{(\ln \frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m}) \sum_{l=1}^{3} \frac{p^l}{|\mathbf{p}|} \mathcal{J}_l^{(j)}})_{s(-s')} \\
&(-1)^{3j+s'} \mathrm{e}^{ip.x} a_{\epsilon'}^*(\mathbf{p}, s'; m, j)) \Bigg) \,.
\end{aligned}
\tag{3.85}
$$

We also have

$$
\begin{aligned}
&{}^{C}\Psi_s^{[0,j]\epsilon}(x) \\
&= (2\pi)^{-\frac{3}{2}} \sum_{s'} \int \mathrm{d}^3\mathbf{p} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (\mathrm{e}^{(\ln \frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m}) \sum_{l=1}^{3} \frac{p^l}{|\mathbf{p}|} \mathcal{J}_l^{(j)}})_{ss'} \\
&(\mathrm{e}^{-ip.x} a_\epsilon(\mathbf{p}, s'; m, j) + (-1)^{3j-s'} \mathrm{e}^{ip.x} a_{\epsilon'}^*(\mathbf{p}, -s'; m, j)).
\end{aligned}
\tag{3.86}
$$

and

$$
\begin{aligned}
&{}^{H}\Psi_s^{[0,j]\epsilon}(x) = (2\pi)^{-\frac{3}{2}} \sum_{s'} \int \mathrm{d}^3\mathbf{p} \Bigg(\frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \big(\mathrm{e}^{-i \arccos \frac{p^3}{|\mathbf{p}|} (-\frac{p^2}{|\mathbf{p}|} \mathcal{J}_1^{(j)} + \frac{p^1}{|\mathbf{p}|} \mathcal{J}_2^{(j)})}\big)_{ss'} \\
&\big(\frac{|\mathbf{p}| + \omega_{\mathbf{p}}}{m}\big)^{s'} \mathrm{e}^{-ip.x} a_\epsilon(\mathbf{p}, s'; m, j) + \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \big(\mathrm{e}^{-i \arccos \frac{p^3}{|\mathbf{p}|} (-\frac{p^2}{|\mathbf{p}|} \mathcal{J}_1^{(j)} + \frac{p^1}{|\mathbf{p}|} \mathcal{J}_2^{(j)})}\big)_{s(-s')} \\
&\big(\frac{|\mathbf{p}| + \omega_{\mathbf{p}}}{m}\big)^{-s'} (-1)^{3j+s'} \mathrm{e}^{-ip.x} a_\epsilon^*(\mathbf{p}, s'; m, j) \Bigg) \,.
\end{aligned}
\tag{3.87}
$$

We also have

$$
\begin{aligned}
&{}^{H}\Psi_s^{[0,j]\epsilon}(x) \\
&= (2\pi)^{-\frac{3}{2}} \sum_{s'} \int \mathrm{d}^3\mathbf{p} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \big(\mathrm{e}^{-i \arccos \frac{p^3}{|\mathbf{p}|} (-\frac{p^2}{|\mathbf{p}|} \mathcal{J}_1^{(j)} + \frac{p^1}{|\mathbf{p}|} \mathcal{J}_2^{(j)})}\big)_{ss'} \\
&\big(\frac{|\mathbf{p}| + \omega_{\mathbf{p}}}{m}\big)^{s'} (\mathrm{e}^{-ip.x} a_\epsilon(\mathbf{p}, s'; m, j) + (-1)^{3j-s'} \mathrm{e}^{ip.x} a_{\epsilon'}^*(\mathbf{p}, -s'; m, j)).
\end{aligned}
\tag{3.88}
$$

In the fermionic case for when $j \in \mathbb{N} + 1/2$ we obtain

$$
\begin{aligned}
&{}^{C}\widetilde{\Psi}_s^{[0,j]\epsilon}(x) = (2\pi)^{-\frac{3}{2}} \sum_{s'} \int \mathrm{d}^3\mathbf{p} \Bigg(\frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (\mathrm{e}^{(\ln \frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m}) \sum_{l=1}^{3} \frac{p^l}{|\mathbf{p}|} \mathcal{J}_l^{(j)}})_{ss'} \\
&\mathrm{e}^{-ip.x} b_\epsilon(\mathbf{p}, s'; m, j) + \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (\mathrm{e}^{(\ln \frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m}) \sum_{l=1}^{3} \frac{p^l}{|\mathbf{p}|} \mathcal{J}_l^{(j)}})_{s(-s')} \\
&(-1)^{3j+s'} \mathrm{e}^{ip.x} b_{\epsilon'}^*(\mathbf{p}, s'; m, j)) \Bigg) \,.
\end{aligned}
\tag{3.89}
$$

We also obtain

$$
\begin{aligned}
&{}^{C}\widetilde{\Psi}_s^{[0,j]\epsilon}(x) \\
&= (2\pi)^{-\frac{3}{2}} \sum_{s'} \int \mathrm{d}^3\mathbf{p} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (\mathrm{e}^{(\ln \frac{|\mathbf{p}|+\omega_{\mathbf{p}}}{m}) \sum_{l=1}^{3} \frac{p^l}{|\mathbf{p}|} \mathcal{J}_l^{(j)}})_{ss'} \\
&(\mathrm{e}^{-ip.x} b_\epsilon(\mathbf{p}, s'; m, j) + (-1)^{3j-s'} \mathrm{e}^{ip.x} b_{\epsilon'}^*(\mathbf{p}, -s'; m, j)).
\end{aligned}
\tag{3.90}
$$

and

$$
{}^{H}\widetilde{\Psi}_s^{[0,j]\epsilon}(x) = (2\pi)^{-\frac{3}{2}} \sum_{s'} \int \mathrm{d}^3\mathbf{p} \Big(\frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \big(\mathrm{e}^{-i \arccos \frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(j)} + \frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(j)})}\big)_{ss'}
$$
$$
\big(\frac{|\mathbf{p}| + \omega_{\mathbf{p}}}{m}\big)^{s'} \mathrm{e}^{-ip.x} b_\epsilon(\mathbf{p}, s'; m, j) + \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \big(\mathrm{e}^{-i \arccos \frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(j)} + \frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(j)})}\big)_{s(-s')}
$$
$$
\big(\frac{|\mathbf{p}| + \omega_{\mathbf{p}}}{m}\big)^{-s'} (-1)^{3j+s'} \mathrm{e}^{-ip.x} b_\epsilon^*(\mathbf{p}, s'; m, j)\Big) \ . \tag{3.91}
$$

and also

$$
\begin{aligned}
&{}^{H}\widetilde{\Psi}_s^{[0,j]\epsilon}(x) \\
&= (2\pi)^{-\frac{3}{2}} \sum_{s'} \int \mathrm{d}^3\mathbf{p} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \big(\mathrm{e}^{-i \arccos \frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(j)} + \frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(j)})}\big)_{ss'} \\
&\big(\frac{|\mathbf{p}| + \omega_{\mathbf{p}}}{m}\big)^{s'} (\mathrm{e}^{-ip.x} b_\epsilon(\mathbf{p}, s'; m, j) + (-1)^{3j-s'} \mathrm{e}^{ip.x} b_{\epsilon'}^*(\mathbf{p}, -s'; m, j)).
\end{aligned} \tag{3.92}
$$

4. Free causal fields for a massless particle of any finite helicity

In this chapter we introduce the construction of free causal fields for massless particles of helicity j by still following the formalism of S.Weinberg in [47],[48] and [51, section 5.9]. Note that the construction of free fields for photons and gravitons is not included in this approach. See [49]. Photons and gravitons are properly associated with potentials instead of fields. The approach that we now follow will be adapted to massless fermions as neutrinos and antineutrinos in the Standard model.

Let $\mathfrak{F}_s^{[j]}$ (resp.$\mathfrak{F}_a^{[j]}$) be the bosonic (resp. fermionic)Fock space for massless bosons(resp.massless fermions) of helicity j. We have

$$
\mathfrak{F}_s^{[j]} = \bigotimes \big(\oplus_{n=0}^{\infty} \otimes_s^n L^2(\mathbb{R}^3)\big) \ . \tag{4.1}
$$

where $\otimes_s^n$ denotes the symmetric n-th tensor product and $\otimes_s^0 L^2(\Sigma_j)$ =$\mathbb{C}$, and

$$
\mathfrak{F}_a^{[j]} = \bigotimes \big(\oplus_{n=0}^{\infty} \otimes_a^n L^2(\mathbb{R}^3)\big) \ . \tag{4.2}
$$

where $\otimes_a^n$ denotes the antisymmetric n-th tensor product and $\otimes_a^0 L^2(\Sigma_j)$ =$\mathbb{C}$.

In the case of N bosons and N fermions with helicities $(j_i)_{1\le i\le N}$ the corresponding Fock spaces, denoted by $\widetilde{\mathfrak{F}}_a^{(N)}$ and $\widetilde{\mathfrak{F}}_s^{(N)}$ respectively, are given by

$$
\widetilde{\mathfrak{F}}_s^{(N)} = \bigotimes_{i=1}^{N} \mathfrak{F}_s^{[j_i]} \tag{4.3}
$$

and

$$
\widetilde{\mathfrak{F}}_a^{(N)} = \bigotimes_{i=1}^{N} \mathfrak{F}_a^{[j_i]} \tag{4.4}
$$

The unitary irreducible representations $\widetilde{U}^{[j]}$ of $\mathcal{P}$ induce two unitary representations of $\mathcal{P}$ in $\mathfrak{F}_s^{[j]}$ and $\mathfrak{F}_a^{[j]}$. Each representation is $\Gamma(\widetilde{U}^{[j]})$.

$a_\epsilon(\mathbf{p},j)$ (resp. $a^*_\epsilon(\mathbf{p},j)$) is the annihilation (resp.creation) operator for a massless boson of helicity j if $\epsilon=+$ and for a antiparticle of helicity j if $\epsilon=-$.

Similarly, $b_\epsilon(\mathbf{p},j)$ (resp. $b^*_\epsilon(\mathbf{p},j)$) is the annihilation (resp.creation) operator for a massless fermion of helicity j if $\epsilon=+$ and for a antiparticle of helicity j if $\epsilon=-$.

The operators $a_\epsilon(\mathbf{p},j)$ and $a^*_\epsilon(\mathbf{p},j)$ fulfil the usual commutation relations (CCR), whereas $b_\epsilon(\mathbf{p},j)$ and $b^*_\epsilon(\mathbf{p},j)$ fulfil the canonical anticommutation relation (CAR). See [51]. Futhermore, the $a's$ commute with the $b's$.

In addition, in the case where several fermions are involved we follow the convention described in [51, sections 4.1 and 4.2]. This means that we will assume that fermionic annihilation and creation operators of different species of particles anticommute for both massive and massless fermions.

Therefore, the following canonical anticommutation and commutation relations hold for a couple of massless particles with helicities j and $j'\neq j$ together with a massive particle with $m>0$ and spin $\widetilde{j}$.

$$\begin{aligned}
&\{b_\epsilon(\mathbf{p},j),b^*_{\epsilon'}(\mathbf{p}',j)\}=\delta_{\epsilon\epsilon'}\delta(\mathbf{p}-\mathbf{p}')\ ,\\
&[a_\epsilon(\mathbf{p},j),a^*_{\epsilon'}(\mathbf{p}',j)]=\delta_{\epsilon\epsilon'}\delta(\mathbf{p}-\mathbf{p}')\ ,\\
&\{b^\sharp_\epsilon(\mathbf{p},j),b^\sharp_{\epsilon'}(\mathbf{p}',j')\}=0\ ,\\
&\{b^\sharp_\epsilon(\xi;m,\widetilde{j}),b^\sharp_{\epsilon'}(\mathbf{p},j')\}=0\ ,\\
&[a^\sharp_\epsilon(\mathbf{p},j),a^\sharp_{\epsilon'}(\mathbf{p}',j')]=0\ ,\\
&[b^\sharp_\epsilon(\mathbf{p},j),a^\sharp_{\epsilon'}(\mathbf{p}',j')]=0\ ,\\
&[b^\sharp_\epsilon(\xi;m,\widetilde{j}),a^\sharp_{\epsilon'}(\mathbf{p}',j)]=0\ .
\end{aligned}\tag{4.5}$$

where $a^\sharp(resp.b^\sharp)$ is $a(resp.b)$ or $a^*(resp.b^*)$.

We now introduce

$$\begin{aligned}
&a_\epsilon(j)(\varphi)=\int_{\mathbb{R}^3}a_\epsilon(\mathbf{p},j)\overline{\varphi(\mathbf{p})}\mathrm{d}^3\mathbf{p}\ ,\quad a^*_\epsilon(j)(\varphi)=\int_{\mathbb{R}^3}a^*_\epsilon(\mathbf{p},j)\varphi(\mathbf{p})\mathrm{d}^3\mathbf{p}\ ,\\
&b_\epsilon(j)(\varphi)=\int_{\mathbb{R}^3}b_\epsilon(\mathbf{p},j)\overline{\varphi(\mathbf{p})}\mathrm{d}^3\mathbf{p}\ ,\quad b^*_\epsilon(j)(\varphi)=\int_{\mathbb{R}^3}b^*_\epsilon(\mathbf{p},j)\varphi(\mathbf{p})\mathrm{d}^3\mathbf{p}\ .
\end{aligned}\tag{4.6}$$

Moreover, for $\varphi\in L^2(\mathbb{R}^3)$, the operators $b_\epsilon(j)$ and $b^*_\epsilon(j)$ are bounded operators on $\mathfrak{F}^j_a$ satisfying

$$\|b^\sharp_\epsilon(j)(\varphi)\|=\|\varphi\|_{L^2}\ .\tag{4.7}$$

From now on we only consider the helicity formalism because it is very useful in Physics.

Furthermore we restrict ourselves to the case of a massless fermion of helicity j and we suppose that the massless fermions we consider are not their own antiparticles.

In that case S.Weinberg (see [47, 48],[49, (2.15),(2.16)] and [51]) has shown that, if we construct a causal field for a massless particle of helicity j by mimicking the construction for a massive particle of spin j, the associated causal field can be constructed only with the annihilation for the massless particle of helicity j and the creation operator for the antiparticle with helicity $-j$. Moreover only the representations (J_1,J_2) of $SL(2,\mathbb{C})$ such that $j=J_2-J_1$ are involved in the construction.

It follows that, if a massless fermion of helicity j is not its own antiparticle, the helicity of the antiparticle is $-j$.

The massless fermion of helicity j is associated to the unitary irreducible representation $\widetilde{U}^{[j]}$ and its antiparticle to the unitary irreducible representation $\widetilde{U}^{[-j]}$.

Let

$$\widetilde{U}^{[|j|]} = \widetilde{U}^{[j]} \oplus \widetilde{U}^{[-j]} \tag{4.8}$$

Let (J_1, J_2) be two spins. For every $M_1 \in (-J_1, -J_1+1, \ldots, J_1-1, J_1)$ and for every $M_2 \in (-J_2, -J_2+1, \ldots, J_2-1, J_2)$ we look for causal free fields, denoted by $\left(\Phi^{[J_1,J_2]\epsilon}_{M_1 M_2}(x)\right)_{M_1 M_2}$, involving particles and antiparticles and satisfying the two fundamental conditions:

(a)The relativistic covariance law:

$$\begin{aligned}&\Gamma(\widetilde{U}^{[|j|]}(A,a))(\Phi^{[J_1,J_2]\epsilon}_{M_1 M_2})(x)\Gamma(\widetilde{U}^{[|j|]}(A,a))^{-1}\\ &= \sum_{M_1' M_2'} D^{[J_1,J_2]}_{M_1 M_2 M_1' M_2'}(A^{-1})(\Phi^{[J_1,J_2]\epsilon}_{M_1' M_2'})(\Lambda(A)x+a)\ ,\end{aligned} \tag{4.9}$$

where $x \in \mathbb{R}^4$.

and

(b)The microscopic causality

$$\{\Phi^{[J_1,J_2]\epsilon}_{M_1 M_2}(x), \Phi^{[J_1,J_2]\epsilon}_{M_1' M_2'}(y)\} = \{\Phi^{[J_1,J_2]\epsilon}_{M_1 M_2}(x), \Phi^{[J_1,J_2]\epsilon,\dagger}_{M_1' M_2'}(y)\} = 0\ . \tag{4.10}$$

for x-y space-like.

As in [48, 3.47] and [51, section 5.9] we set

$$\begin{aligned}&(\Phi^{[J_1,J_2]\epsilon}_{M_1 M_2})(x)\\ &= (\frac{1}{2\pi})^{\frac{3}{2}} \int \mathrm{d}^3\mathbf{p}\big(\alpha(u^{[J_1,J_2]}_{M_1 M_2})(\mathbf{p}, j)\mathrm{e}^{-ip.x} b_\epsilon(\mathbf{p}, j)\\ &\qquad + \beta(v^{[J_1,J_2]}_{M_1 M_2})(\mathbf{p}, -j)\mathrm{e}^{ip.x} b^*_{\epsilon'}(\mathbf{p}, -j)\big)\ .\end{aligned} \tag{4.11}$$

where $\epsilon \neq \epsilon'$.

We now study the transformation rules of the annihilation and creation operators by $\Gamma(\widetilde{U}^{[|j|]})$. By [48] and [51, section 5.9] we easily get

$$\begin{aligned}&\Gamma(\widetilde{U}^{[j]}(A,a)\oplus I)b_\epsilon(\mathbf{p}\ , j)\Gamma(\widetilde{U}^{[j]}(A,a)\oplus I)^{-1}\\ &= \Big(\frac{|\mathbf{p}_{\Lambda(A)p}|}{|\mathbf{p}|}\Big)^{\frac{1}{2}} \mathrm{e}^{-ia.\Lambda(A)p} L^{-j}\big((A^2_{\Lambda(A)p})^{-1} A A^2_p\big) b_\epsilon(\mathbf{p}_{\Lambda(A)p}\ , j)\ .\end{aligned} \tag{4.12}$$

$$\begin{aligned}&\Gamma(\widetilde{U}^{[j]}(A,a)\oplus I)b^*_\epsilon(\mathbf{p}\ , j)\Gamma(\widetilde{U}^{[j]}(A,a)\oplus I)^{-1}\\ &= \Big(\frac{|\mathbf{p}_{\Lambda(A)p}|}{|\mathbf{p}|}\Big)^{\frac{1}{2}} \mathrm{e}^{ia.\Lambda(A)p} L^{j}\big((A^2_{\Lambda(A)p})^{-1} A A^2_p\big) b^*_\epsilon(\mathbf{p}_{\Lambda(A)p}\ , j)\ .\end{aligned} \tag{4.13}$$

$$\begin{aligned}&\Gamma(I\oplus\widetilde{U}^{[-j]}(A,a))b^*_{\epsilon'}(\mathbf{p}\ , -j)\Gamma(I\oplus\widetilde{U}^{[-j]}(A,a))^{-1}\\ &= \Big(\frac{|\mathbf{p}_{\Lambda(A)p}|}{|\mathbf{p}|}\Big)^{\frac{1}{2}} \mathrm{e}^{ia.\Lambda(A)p} L^{-j}\big((A^2_{\Lambda(A)p})^{-1} A A^2_p\big) b^*_{\epsilon'}(\mathbf{p}_{\Lambda(A)p}\ , -j)\ .\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}&\Gamma(I\oplus\widetilde{U}^{[-j]}(A,a))b_{\epsilon'}(\mathbf{p}\ , -j)\Gamma(I\oplus\widetilde{U}^{[-j]}(A,a))^{-1}\\ &= \Big(\frac{|\mathbf{p}_{\Lambda(A)p}|}{|\mathbf{p}|}\Big)^{\frac{1}{2}} \mathrm{e}^{-ia.\Lambda(A)p} L^{j}\big((A^2_{\Lambda(A)p})^{-1} A A^2_p\big) b_{\epsilon'}(\mathbf{p}_{\Lambda(A)p}\ , -j)\ .\end{aligned} \tag{4.15}$$

From now on we omit the superscript $[J_1, J_2]$. We shall introduce it again later on. By (4.9), (4.11) and (4.12) we obtain, for $A \in SL(2,\mathbb{C})$,

$$\begin{aligned}&\sum_{M_1' M_2'} \Big(\frac{|\mathbf{p}|}{|\mathbf{p}_{\Lambda(A)p}|}\Big)^{\frac{1}{2}} D_{M_1 M_2 M_1' M_2'}(A) u_{M_1' M_2'}(\mathbf{p}\ , j)\\ &= L^j\big((A^2_{\Lambda(A)p})^{-1} A A_p^2\big) u_{M_1 M_2}(\mathbf{p}_{\Lambda(A)p}\ , j)\ .\end{aligned} \tag{4.16}$$

and by (4.9), (4.11) and (4.14) we get

$$\begin{aligned}&\sum_{M_1' M_2'} \Big(\frac{|\mathbf{p}|}{|\mathbf{p}_{\Lambda(A)p}|}\Big)^{\frac{1}{2}} D_{M_1 M_2 M_1' M_2'}(A) v_{M_1' M_2'}(\mathbf{p}\ , -j)\\ &= L^j\big((A^2_{\Lambda(A)p})^{-1} A A_p^2\big) v_{M_1 M_2}(\mathbf{p}_{\Lambda(A)p}\ , -j)\ .\end{aligned} \tag{4.17}$$

Setting $p{=}k_0$ we then get

$$\begin{aligned}u_{M_1 M_2}(\mathbf{p}, j) &= (|\mathbf{p}|)^{-\frac{1}{2}} \sum_{M_1' M_2'} D_{M_1 M_2 M_1' M_2'}(A_p^2) u_{M_1' M_2'}(\mathbf{k}_0, j)\ ,\\ v_{M_1 M_2}(\mathbf{p}, -j) &= (|\mathbf{p}|)^{-\frac{1}{2}} \sum_{M_1' M_2'} D_{M_1 M_2 M_1' M_2'}(A_p^2) v_{M_1' M_2'}(\mathbf{k}_0, -j)\ .\end{aligned} \tag{4.18}$$

Recall that A_p^2 is given by (2.44) .

4.1. Computation of $u_{M_1 M_2}(\mathbf{k}_0, j)$ and $v_{M_1 M_2}(\mathbf{k}_0, -j)$.

Let A_φ be the following rotation

$$A_\varphi = \begin{pmatrix} \mathrm{e}^{-i\frac{\varphi}{2}} & 0 \\ 0 & \mathrm{e}^{i\frac{\varphi}{2}} \end{pmatrix} \tag{4.19}$$

We have

$$\begin{aligned}&\Lambda(A_\varphi) k_0 = k_0\ ,\\ &(A^2_{\Lambda(A_\varphi)k_0})^{-1} A_\varphi A^2_{k_0} = A_\varphi\ ,\\ &A_\varphi = \mathrm{e}^{-i\varphi\frac{\sigma_3}{2}}\ .\end{aligned} \tag{4.20}$$

Combining this with (3.27), (3.28), (3.31), (3.34), (3.37), (4.16) and (4.17) we easily get

$$\begin{aligned}\mathrm{e}^{-ij\varphi} u_{M_1 M_2}(\mathbf{k}_0, j) &= \mathrm{e}^{-i\varphi(M_1+M_2)} u_{M_1 M_2}(\mathbf{k}_0, j)\ ,\\ \mathrm{e}^{-ij\varphi} v_{M_1 M_2}(\mathbf{k}_0, -j) &= \mathrm{e}^{-i\varphi(M_1+M_2)} v_{M_1 M_2}(\mathbf{k}_0, -j)\ .\end{aligned} \tag{4.21}$$

This proves that $u_{M_1 M_2}(\mathbf{k}_0, j)$ and $v_{M_1 M_2}(\mathbf{k}_0, -j)$ are different from zero if and only if

$$M_1 + M_2 = j \tag{4.22}$$

Let A_z be the following transformation

$$A_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \tag{4.23}$$

We have

$$\begin{aligned}&\Lambda(A_z)k_0 = k_0\ ,\\&(A^2_{\Lambda(A_z)k_0})^{-1}A_zA^2_{k_0} = A_z\ .\end{aligned}\tag{4.24}$$

We get, for $z=\lambda+i\mu$,

$$\Lambda(A_z)=\begin{pmatrix}1+\frac{|z|^2}{2} & \lambda & -\mu & -\frac{|z|^2}{2}\\ \lambda & 1 & 0 & -\lambda\\ -\mu & 0 & 1 & \mu\\ \frac{|z|^2}{2} & \lambda & -\mu & 1-\frac{|z|^2}{2}\end{pmatrix}\tag{4.25}$$

By (3.27) A_z is the transformation

$$\mathrm{e}^{-i\left(\lambda(\mathrm{M}_{10}+\mathrm{M}_{13})-\mu(\mathrm{M}_{20}+\mathrm{M}_{23})\right)}\tag{4.26}$$

Here M_{10}, M_{13}, M_{20} and M_{23} are given in (3.28).
This yields

$$A_z=\mathrm{e}^{i\left(\lambda(\frac{\sigma_2}{2}-i\frac{\sigma_1}{2})+\mu(\frac{\sigma_1}{2}+i\frac{\sigma_2}{2})\right)}\tag{4.27}$$

It follows from (2.37), (4.16), (4.17) and (4.24) that

$$\begin{aligned}&u_{M_1M_2}(\mathbf{k}_0,j)=\sum_{M_1'M_2'}D^{[J_1,J_2]}_{M_1M_2M_1'M_2'}(A_z)u_{M_1'M_2'}(\mathbf{k}_0,j)\ ,\\&v_{M_1M_2}(\mathbf{k}_0,-j)=\sum_{M_1'M_2'}D^{[J_1,J_2]}_{M_1M_2M_1'M_2'}(A_z)v_{M_1'M_2'}(\mathbf{k}_0,-j)\ .\end{aligned}\tag{4.28}$$

By (3.24) and (3.32) we have in the representation associated with $D^{[J_1,J_2]}$

$$\begin{aligned}&\mathrm{M}_{10}+\mathrm{M}_{13}=-i(\mathcal{A}_1-\mathcal{B}_1)-(\mathcal{A}_2+\mathcal{B}_2)\ ,\\&\mathrm{M}_{20}+\mathrm{M}_{23}=-i(\mathcal{A}_2-\mathcal{B}_2)+(\mathcal{A}_1+\mathcal{B}_1)\ .\end{aligned}\tag{4.29}$$

By (4.26), (4.28) and (4.29) we have

$$\sum_{M_1'M_2'}\big(-i(\mathcal{A}_1-\mathcal{B}_1)-(\mathcal{A}_2+\mathcal{B}_2)\big)_{M_1M_2M_1'M_2'}u_{M_1'M_2'}(\mathbf{k}_0,j)=0\tag{4.30}$$

$$\sum_{M_1'M_2'}\big(-i(\mathcal{A}_2-\mathcal{B}_2)+(\mathcal{A}_1+\mathcal{B}_1)\big)_{M_1M_2M_1'M_2'}u_{M_1'M_2'}(\mathbf{k}_0,j)=0\tag{4.31}$$

By (3.34) and (3.35) we get from (4.30) and (4.31)

$$\begin{aligned}&\sum_{M_1'}\big((\mathcal{J}_2^{(1)}+i\mathcal{J}_1^{(1)})_{M_1M_1'}u_{M_1'M_2}(\mathbf{k}_0,j)+\\&\sum_{M_2'}(\mathcal{J}_2^{(2)}-i\mathcal{J}_1^{(2)})_{M_2M_2'}u_{M_1M_2'}(\mathbf{k}_0,j)\big)=0\ .\end{aligned}\tag{4.32}$$

$$\begin{aligned}&\sum_{M_1'}\big((-\mathcal{J}_1^{(1)}+i\mathcal{J}_2^{(1)})_{M_1M_1'}u_{M_1'M_2}(\mathbf{k}_0,j)+\\&\sum_{M_2'}(-\mathcal{J}_1^{(2)}-i\mathcal{J}_2^{(2)})_{M_2M_2'}u_{M_1M_2'}(\mathbf{k}_0,j)\big)=0\ .\end{aligned}\tag{4.33}$$

It follows from (4.32) and (4.33) that

$$\sum_{M_1'}(\mathcal{J}_1^{(1)} - i\mathcal{J}_2^{(1)})_{M_1M_1'}u_{M_1'M_2}(\mathbf{k}_0, j) = 0 \ . \tag{4.34}$$

$$\sum_{M_2'}(\mathcal{J}_1^{(2)} + i\mathcal{J}_2^{(2)})_{M_2M_2'}u_{M_1M_2'}(\mathbf{k}_0, j) = 0 \ . \tag{4.35}$$

In view of (3.34), (4.34) and (4.35) $u_{M_1M_2}(\mathbf{k}_0, j)$ is equal to zero unless

$$M_1 = -J_1, M_2 = J_2 \ . \tag{4.36}$$

By (4.28) the same is true for $v_{M_1'M_2'}(\mathbf{k}_0, -j)$ and by (4.22) we must have

$$j = J_2 - J_1 \tag{4.37}$$

We finally set by applying the normalization used in Physics

$$u_{M_1M_2}(\mathbf{k}_0, j) = v_{M_1'M_2'}(\mathbf{k}_0, -j) = \delta_{M_1,-J_1}\delta_{M_2,J_2}2^{J_1+J_2-1/2} \tag{4.38}$$

This, together with (4.18), yields

$$u_{M_1M_2}(\mathbf{p}, j) = v_{M_1M_2}(\mathbf{p}, -j) = (2|\mathbf{p}|)^{-\frac{1}{2}}D^{[J_1,J_2]}_{M_1M_2-J_1J_2}(A_p^2) \ . \tag{4.39}$$

In view of (2.43) and (2.44) we obtain in the representation $D^{[J_1,J_2]}(\boldsymbol{.})$

$$D^{[J_1,J_2]}(A_p^2) = D^{[J_1,J_2]}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}}\mathrm{e}^{-i\ln|\mathbf{p}|(\widetilde{K_3})}) \tag{4.40}$$

where $B_{\frac{\mathbf{p}}{|\mathbf{p}|}}$ is given by (3.40) .

This, together with (3.34), (3.36) and (3.37), yields

$$D^{[J_1,J_2]}_{M_1M_2(-J_1)J_2}(A_p^2) = |\mathbf{p}|^{J_1+J_2}(D^{J_1}_{M_1(-J_1)}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})D^{J_2}_{M_2J_2}(B_{\frac{\mathbf{p}}{|\mathbf{p}|}})) \ . \tag{4.41}$$

Combining this with (3.44) and (4.39) we then get

$$\begin{aligned} &u^{[J_1,J_2]}_{M_1M_2}(\mathbf{p}, j) = v^{[J_1,J_2]}_{M_1M_2}(\mathbf{p}, -j) = (2|\mathbf{p}|)^{J_1+J_2-1/2} \\ &\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(1)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(1)})}\big)_{M_1(-J_1)}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(2)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(2)})}\big)_{M_2J_2} \ . \end{aligned} \tag{4.42}$$

In [47] two particular cases are considered. For a left-handed particle with helicity $j < 0$ one can choose $J_2 = 0$ and $J_1 = -j = |j|$ and we have

$$u_s^{[-j,0]+}(\mathbf{p}, j) = (2|\mathbf{p}|)^{|j|-1/2}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(-j)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(-j)})}\big)_{sj} \ . \tag{4.43}$$

where $s = (-|j|, -|j|+1, \cdots, |j|-1, |j|)$.

For a right-handed particle with helicity $j > 0$ one can choose $J_1 = 0$ and $J_2 = j$. We then get

$$u_s^{[0,j]+}(\mathbf{p}, j) = (2|\mathbf{p}|)^{j-1/2}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(j)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(j)})}\big)_{sj} \ . \tag{4.44}$$

where $s = (-j, -j+1, \cdots, j-1, j)$.

This gives for a neutrino

$$u_s^{[-1/2,0]+}(\mathbf{p}, -1/2) = \big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(1/2)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(1/2)})}\big)_{s(-1/2)} \ . \tag{4.45}$$

where $s = -1/2, 1/2$. and for an antineutrino

$$u_s^{[0,1/2]+}(\mathbf{p}, 1/2) = \big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}_1^{(1/2)}+\frac{p^1}{|\mathbf{p}|}\mathcal{J}_2^{(1/2)})}\big)_{s(1/2)} \ . \tag{4.46}$$

where $s = -1/2, 1/2$.

In order to satisfy the microscopic condition (4.10) with $u^{[J_1,J_2]}_{M_1M_2}(\mathbf{p},j)$ and $v^{[J_1,J_2]}_{M_1M_2}(\mathbf{p},-j)$ given by (4.42) S.Weinberg has shown in [51, section 5.9] that we must have $|\alpha| = |\beta|$ and that we can choose $\alpha = \beta$.

Thus, up to an over-all scale of the field, we finally get

$$\begin{aligned}
&(\Phi^{[J_1,J_2]\epsilon}_{M_1M_2})(x) = (\frac{1}{2\pi})^{\frac{3}{2}}\int \mathrm{d}^3\mathbf{p}(2|\mathbf{p}|)^{J_1+J_2-1/2}\\
&\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}^{(1)}_1+\frac{p^1}{|\mathbf{p}|}\mathcal{J}^{(1)}_2)}\big)_{M_1(-J_1)}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}^{(2)}_1+\frac{p^1}{|\mathbf{p}|}\mathcal{J}^{(2)}_2)}\big)_{M_2J_2}\\
&\big(\mathrm{e}^{-ip.x}b_\epsilon(\mathbf{p},j)+\mathrm{e}^{ip.x}b^*_{\epsilon'}(\mathbf{p},-j)\big)\ .
\end{aligned}\tag{4.47}$$

where $J_2 - J_1 = j$.

For a left-handed particle of helicity $j < 0$ we get

$$\begin{aligned}
&(\Phi^{[-j,0]+}_s)(x) = (\frac{1}{2\pi})^{\frac{3}{2}}\int \mathrm{d}^3\mathbf{p}(2|\mathbf{p}|)^{-j-1/2}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}^{(-j)}_1+\frac{p^1}{|\mathbf{p}|}\mathcal{J}^{(-j)}_2)}\big)_{sj}\\
&\big(\mathrm{e}^{-ip.x}b_+(\mathbf{p},j)+\mathrm{e}^{ip.x}b^*_-(\mathbf{p},-j)\big)\ .
\end{aligned}\tag{4.48}$$

where $s = (-|j|, -|j|+1, \cdots, |j|-1, |j|)$

For a right- handed particle of helicity $j > 0$ we obtain

$$\begin{aligned}
&(\Phi^{[0,j]+}_s)(x) = (\frac{1}{2\pi})^{\frac{3}{2}}\int \mathrm{d}^3\mathbf{p}(2|\mathbf{p}|)^{j-1/2}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\mathcal{J}^{(j)}_1+\frac{p^1}{|\mathbf{p}|}\mathcal{J}^{(j)}_2)}\big)_{sj}\\
&\big(\mathrm{e}^{-ip.x}b_+(\mathbf{p},j)+\mathrm{e}^{ip.x}b^*_-(\mathbf{p},-j)\big)\ .
\end{aligned}\tag{4.49}$$

where $s = (-j, -j+1, \cdots, j-1, j)$

For a neutrino we get

$$\begin{aligned}
&(\Phi^{[-1/2,0]+}_s)(x) = (\frac{1}{2\pi})^{\frac{3}{2}}\int \mathrm{d}^3\mathbf{p}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\frac{\sigma_1}{2}+\frac{p^1}{|\mathbf{p}|}\frac{\sigma_2}{2})}\big)_{s(-1/2)}\\
&\big(\mathrm{e}^{-ip.x}b_+(\mathbf{p},-1/2)+\mathrm{e}^{ip.x}b^*_-(\mathbf{p},1/2)\big)\ .
\end{aligned}\tag{4.50}$$

and for an antineutrino we obtain

$$\begin{aligned}
&(\Phi^{[0,1/2]+}_s)(x) = (\frac{1}{2\pi})^{\frac{3}{2}}\int \mathrm{d}^3\mathbf{p}\big(\mathrm{e}^{-i\arccos\frac{p^3}{|\mathbf{p}|}(-\frac{p^2}{|\mathbf{p}|}\frac{\sigma_1}{2}+\frac{p^1}{|\mathbf{p}|}\frac{\sigma_2}{2})}\big)_{s(1/2)}\\
&\big(\mathrm{e}^{-ip.x}b_+(\mathbf{p},1/2)+\mathrm{e}^{ip.x}b^*_-(\mathbf{p},-1/2)\big)\ .
\end{aligned}\tag{4.51}$$

Here $s = (1/2, -1/2)$.

5. Definition of the model

We consider a model which is a generalization of the weak decay of the nucleus ${}^{60}_{27}Co$ into the nucleus ${}^{60}_{28}Ni^*$, e^- and $\overline{\nu}_e$.

$${}^{60}_{27}Co \rightarrow^{60}_{28} Ni^* + e^- + \overline{\nu}_e \tag{5.1}$$

Spin(${}^{60}_{27}Co$) = 5 and Spin(${}^{60}_{28}Ni^*$) = 4. In this decay parity is not conserved.

Our model involves four particles : two bosons of mass $m_1 > 0$ and spin j_1 and of mass $m_2 > 0$ and spin j_2 respectively, a fermion of mass $m_3 > 0$ and spin j_3 and a massless fermion of helicity $-j_4$ which is the antiparticle of a massless fermion of helicity $j_4 < 0$ as it follows from the conservation of the leptonic number.

Set $\xi_i = (\mathbf{p}_i, s_i)$ for each $i = 1, 2, 3$, i.e., for the massive bosons and fermion.We have, for each $i = 1, 2, 3$, $\int \mathrm{d}\xi_i = \sum_{s_i} \int \mathrm{d}^3\mathbf{p}_i$.

For the massless fermion we set $\xi_4 = (\mathbf{p}_4, j_4)$ and $\widetilde{\xi_4} = (\mathbf{p}_4, -j_4)$ with $\int \mathrm{d}\xi_4 = \int \mathrm{d}^3\mathbf{p}_4$.

The Fock space of the system is

$$\mathfrak{F} = \mathfrak{F}_s^{[m_1,j_1]} \otimes \mathfrak{F}_s^{[m_2,j_2]} \otimes \mathfrak{F}_a^{[m_3,j_3]} \otimes \mathfrak{F}_a^{[-j_4]} \tag{5.2}$$

Ω shall denote the vacuum in $\mathfrak{F}$.

The free Hamiltonian H_0 is given by

$$H_0 = \sum_{i=1}^{2} \int w^i(\xi_i) a_+^*(\xi_i; m_i, j_i) a_+(\xi_i; m_i, j_i) \mathrm{d}\xi_i + \int w^3(\xi_3) b_+^*(\xi_3; m_3, j_3) b_+(\xi_3; m_3, j_3) \mathrm{d}\xi_3$$
$$+ \int w^4(\xi_4) b_-^*(\widetilde{\xi_4}) b_-(\widetilde{\xi_4}) \mathrm{d}\xi_4 \tag{5.3}$$

The free relativistic energies of the massive bosons and fermion and of the massless fermion are given by

$$w^i(\xi_i) = (|\mathbf{p}_i|^2 + m_i^2)^{1/2}, i = 1, 2, 3 \tag{5.4}$$
$$w^4(\xi_4) = |\mathbf{p}_4| \tag{5.5}$$

From now on we suppose that

$$\begin{aligned} m_1 &> m_2 > m_3 \\ m_1 &> m_2 + m_3 \ . \end{aligned} \tag{5.6}$$

H_0 is a self-adjoint operator in $\mathfrak{F}$.

In the interaction representation the formal interaction,denoted by $H_I(t)$, is given by

$$H_I(t) = \int \mathrm{d}^3\mathbf{x} \mathcal{H}(t, \mathbf{x}) \tag{5.7}$$

The S-matrix will be Lorentz-invariant if

$$\Gamma(U(A,a)) \mathcal{H}(x) \Gamma(U(A,a))^{-1} = \mathcal{H}(\Lambda(A)x + a) \tag{5.8}$$
$$[\mathcal{H}(x), \mathcal{H}(y)] = 0, (x-y)^2 \le 0 \ . \tag{5.9}$$

The general form of of $\mathcal{H}(x)$ in terms of the causal free fields is given in [51, (5.1.9) and (5.1.10)].

By (3.63) and (3.65) we get for each $i = 1, 2, 3$

$$\begin{aligned} &({}^C u^{[J_1^i, J_2^i]}_{M_1^i M_2^i})(\xi_i; m_i, j_i) \\ &\quad = (2\pi)^{-3/2} \frac{1}{\sqrt{2\omega_{\mathbf{p}_i}}} \sum_{M_1^{i\prime} M_2^{i\prime}} \big(\mathrm{e}^{-(\ln \frac{|\mathbf{p}_i| + \omega_{\mathbf{p}_i}}{m}) \sum_{l=1}^3 \frac{p_i^l}{|\mathbf{p}_i|} \mathcal{J}_l^{(i,1)}}\big)_{M_1^i M_1^{i\prime}} \\ &\qquad (\mathrm{e}^{\ln \frac{|\mathbf{p}_i| + \omega_{\mathbf{p}_i}}{m}) \sum_{l=1}^3 \frac{p_i^l}{|\mathbf{p}_i|} \mathcal{J}_l^{(i,2)}})_{M_2^i M_2^{i\prime}} . (J_1^i J_2^i j_i s_i | J_1^i M_1^i J_2^i M_2^{i\prime})\big). \end{aligned} \tag{5.10}$$

$$({}^{H}u^{[J_1^i,J_2^i]}_{M_1^iM_2^i})(\xi_i;m_i,j_i)=(2\pi)^{-3/2}$$
$$\frac{1}{\sqrt{2\omega_{\mathbf{p}_i}}}\Big(\sum_{M_1^{i\prime}M_2^{i\prime}}(\frac{|\mathbf{p}_i|+\omega_{\mathbf{p}_i}}{m})^{M_2^{i\prime}-M_1^{i\prime}}\big(\mathrm{e}^{-i\arccos\frac{p_i^3}{|\mathbf{p}_i|}(-\frac{p_i^2}{|\mathbf{p}_i|}\mathcal{J}_1^{(i,1)}+\frac{p_i^1}{|\mathbf{p}_i|}\mathcal{J}_2^{(i,1)})}\big)_{M_1^iM_1^{i\prime}}$$
$$\big(\mathrm{e}^{-i\arccos\frac{p_i^3}{|\mathbf{p}_i|}(-\frac{p_i^2}{|\mathbf{p}_i|}\mathcal{J}_1^{(i,2)}+\frac{p_i^1}{|\mathbf{p}_i|}\mathcal{J}_2^{(i,2)})}\big)_{M_2^iM_2^{i\prime}}(J_1^iJ_2^ij_is_i|J_1^iM_1^{i\prime}J_2^iM_2^{i\prime})\Big). \quad (5.11)$$

Here $J_{\boldsymbol{.}}^i$ and $M_{\boldsymbol{.}}^i$ are associated to the spin of the particle i. $\mathcal{J}_{\boldsymbol{.}}^{(i,.)}$ are the generators of the rotations in the representation $D^{J_{\boldsymbol{.}}^i}(\boldsymbol{.})$.

For the massless fermion we only consider the helicity formalism and, by (4.42), we set

$$u^{[J_1^4,J_2^4]}_{M_1^4M_2^4}(\xi_4)=(2|\mathbf{p}_4|)^{J_1^4+J_2^4-1/2}\big(\mathrm{e}^{-i\arccos\frac{p_4^3}{|\mathbf{p}_4|}(-\frac{p_4^2}{|\mathbf{p}_4|}\mathcal{J}_1^{(4,1)}+\frac{p_4^1}{|\mathbf{p}_4|}\mathcal{J}_2^{(2,1)})}\big)_{M_1^2(-J_1^2)}$$
$$\big(\mathrm{e}^{-i\arccos\frac{p_2^3}{|\mathbf{p}_2|}(-\frac{p_2^2}{|\mathbf{p}_2|}\mathcal{J}_1^{(2,2)}+\frac{p_4^1}{|\mathbf{p}_4|}\mathcal{J}_2^{(4,2)})}\big)_{M_2^4J_2^4}. \quad (5.12)$$

where $J_{\boldsymbol{.}}^4$ and $M_{\boldsymbol{.}}^4$ are associated to the spin of the massless fermion.
$\mathcal{J}_{\boldsymbol{.}}^{(4,.)}$ are the generators of the translations in the representation $D^{J_{\boldsymbol{.}}^4}(\boldsymbol{.})$.
By (3.70) we now set for the massive bosons, $i=1,2$,

$$ {}_1^{\sharp}\Phi^{[J_1^i,J_2^i]}_{M_1^iM_2^i}(x)=(2\pi)^{-\frac{3}{2}}\int \mathrm{d}\xi_i({}^{\sharp}u^{[J_1^i,J_2^i]}_{M_1^iM_2^i})(\xi_i;m_i,j_i)\mathrm{e}^{-ip_3.x}a_+(\xi_i;m_i,j_i). \quad (5.13)$$

and, by (3.76), for the massive fermion

$$ {}_1^{\sharp}\Psi^{[J_1^3,J_2^3]}_{M_1^3M_2^3}(x)=(2\pi)^{-\frac{3}{2}}\int \mathrm{d}\xi_3({}^{\sharp}u^{[J_1^3,J_2^3]}_{M_1^3M_2^3})(\xi_3;m_3,j_3)\mathrm{e}^{-ip_3.x}b_+(\xi_3;m_3,j_3). \quad (5.14)$$

Finally, by (4.47), for the massless fermion we let

$$({}_2\Psi^{[J_1^4,J_2^4]}_{M_1^4M_2^4})(x)=(\frac{1}{2\pi})^{\frac{3}{2}}\int \mathrm{d}\xi_4 u^{[J_1^4,J_2^4]}_{M_1^4M_2^4}(\xi_4)\mathrm{e}^{ip_4.x}b_-^*(\widetilde{\xi_4}). \quad (5.15)$$

Let us now write down the formal interaction,denoted by V_I, of the three particles and antiparticles in the Schrödinger representation.We have

$$V_I=\big(V_I^{(1)}+V_I^{(2)}+\widetilde{V}_I^{(1)}+\widetilde{V}_I^{(2)}\big) \quad (5.16)$$

$V_I^{(1)}$ is given by

$$V_I^{(1)}=\int \mathrm{d}^3\mathbf{x}\sum_{M_1^1M_2^1M_1^2M_2^2M_1^3M_2^3M_1^4M_2^4}\big(g^{(1)}_{M_1^1M_2^1M_1^2M_2^2M_1^3M_2^3M_1^4M_2^4}$$
$$\big({}_2\Psi^{[J_1^4,J_2^4]}_{M_1^4M_2^4}\big)(0,\mathbf{x})\big({}_1^{\sharp}\Psi^{[J_1^3,J_2^3]}_{M_1^3M_2^3}\big)^*(0,\mathbf{x})\big({}_1^{\sharp}\Phi^{[J_1^2,J_2^2]}_{M_1^2M_2^2}\big)^*(0,\mathbf{x})\big({}_1^{\sharp}\Phi^{[J_1^1,J_2^1]}_{M_1^1M_2^1}\big)(0,\mathbf{x})\big). \quad (5.17)$$

$V_I^{(2)}$ is given by

$$V_I^{(2)}=\int \mathrm{d}^3\mathbf{x}\sum_{M_1^1M_2^1M_1^2M_2^2M_1^3M_2^3M_1^4M_2^4}\overline{\big(g^{(1)}_{M_1^1M_2^1M_1^2M_2^2M_1^3M_2^3M_1^4M_2^4}}$$
$$\big({}_1^{\sharp}\Phi^{[J_1^1,J_2^1]}_{M_1^1M_2^1}\big)^*(0,\mathbf{x})\big({}_1^{\sharp}\Phi^{[J_1^2,J_2^2]}_{M_1^2M_2^2}\big)(0,\mathbf{x})\big({}_1^{\sharp}\Psi^{[J_1^3,J_2^3]}_{M_1^3M_2^3}\big)(0,\mathbf{x})\big)\big({}_2\Psi^{[J_1^4,J_2^4]}_{M_1^4M_2^4}\big)^*(0,\mathbf{x})\big). \quad (5.18)$$

and we have

$$\widetilde{V}_I^{(1)} = \int \mathrm{d}^3\mathbf{x} \sum_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} \big(g^{(2)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4}$$
$$\big({}_2\Psi^{[J_1^4,J_2^4]}_{M_1^4 M_2^4}\big)(0,\mathbf{x})\big({}^\sharp_1\Psi^{[J_1^3,J_2^3]}_{M_1^3 M_2^3}\big)^*(0,\mathbf{x})\big({}^\sharp_1\Phi^{[J_1^2,J_2^2]}_{M_1^2 M_2^2}\big)^*(0,\mathbf{x})\big({}^\sharp_1\Phi^{[J_1^1,J_2^1]}_{M_1^1 M_2^1}\big)^*(0,\mathbf{x})\big). \tag{5.19}$$

and

$$\widetilde{V}_I^{(2)} = \int \mathrm{d}^3\mathbf{x} \sum_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} \overline{\big(g^{(2)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4}}$$
$$\big({}^\sharp_1\Phi^{[J_1^1,J_2^1]}_{M_1^1 M_2^1}\big)(0,\mathbf{x})\big({}^\sharp_1\Phi^{[J_1^2,J_2^2]}_{M_1^2 M_2^2}\big)(0,\mathbf{x})\big({}^\sharp_1\Psi^{[J_1^3,J_2^3]}_{M_1^3 M_2^3}\big)(0,\mathbf{x})\big)\big({}_2\Psi^{[J_1^4,J_2^4]}_{M_1^4 M_2^4}\big)^*(0,\mathbf{x})\big). \tag{5.20}$$

V_I is formally self adjoint.

By [51, 5.1.10] the constants $g^{(i)}_{\cdot}$,$i = 1, 2$, have to satisfy the following condition for $i = 1, 2$, and for every $A \in SL(2, \mathbb{C})$

$$g^{(i)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} =$$
$$\sum_{M_1^{1\prime} M_2^{1\prime} M_1^{2\prime} M_2^{2\prime} M_1^{3\prime} M_2^{3\prime} M_1^{4\prime} M_2^{4\prime}} D^{[J_1^1,J_2^1]}_{M_1^{1\prime} M_2^{1\prime} M_1^1 M_2^1}(A^{-1}) D^{[J_1^2,J_2^2]}_{M_1^{2\prime} M_2^{2\prime} M_1^2 M_2^2}(A^{-1})$$
$$D^{[J_1^3,J_2^3]}_{M_1^{3\prime} M_2^{3\prime} M_1^3 M_2^3}(A^{-1}) D^{[J_1^4,J_2^4]}_{M_1^{4\prime} M_2^{4\prime} M_1^4 M_2^4}(A^{-1}) g^{(i)}_{M_1^{1\prime} M_2^{1\prime} M_1^{2\prime} M_2^{2\prime} M_1^{3\prime} M_2^{3\prime} M_1^{4\prime} M_2^{4\prime}}. \tag{5.21}$$

The coefficients $g^{(i)}_{\cdot}$,$i = 1, 2$, are associated with the coupling of the spins J_1^1,J_1^2, J_1^3 and J_1^4 and with the coupling of the spins J_2^1,J_2^2, J_2^3 and J_1^3 to make scalars.See [51, section5],[46] and [50].

After integrating with respect to $\mathbf{x}$ we obtain

$$V_I^{(1)} =$$
$$(2\pi)^{-3} \sum_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} g^{(1)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4$$
$$\delta^3(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$
$$\big(u^{[J_1^4,J_2^4]}_{M_1^4 M_2^4}(\xi_4) b_-^*(\widetilde{\xi_4}) \overline{{}^\sharp u^{[J_1^3,J_2^3]}_{M_1^3 M_2^3})(\xi_3; m_3, j_3)} b_+^*(\xi_3; m_3, j_3)\big) \times$$
$$\overline{({}^\sharp u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2})(\xi_2; m_2, j_2)} a_+^*(\xi_2; m_2, j_2) {}^\sharp u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1})(\xi_1; m_1, j_1) a_+(\xi_1; m_1, j_1)\big) \ . \tag{5.22}$$

and

$$V_I^{(2)} =$$
$$(2\pi)^{-3} \sum_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} \overline{g^{(1)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4}} \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4$$
$$\delta^3(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$
$$\overline{\big({}^\sharp u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1}(\xi_1; m_1, j_1)} a_+^*(\xi_1; m_1, j_1) {}^\sharp u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2}(\xi_2; m_2, j_2) a_+(\xi_2; m_2, j_2)\big) \times$$
$$\big({}^\sharp u^{[J_1^3,J_2^3]}_{M_1^3 M_2^3}(\xi_3; m_3, j_3) b_+(\xi_3; m_3, j_3) \overline{{}^\sharp u^{[J_1^4,J_2^4]}_{M_1^4 M_2}}(\xi_4) b_-(\widetilde{\xi_4})\big) \ . \tag{5.23}$$

together with

$$
\begin{aligned}
&\widetilde{V}_I^{(1)} = \\
&\quad (2\pi)^{-3} \sum_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} g^{(2)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 \\
&\qquad \delta^3(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\
&\qquad \big(u^{[J_1^4,J_2^4]}_{M_1^4 M_2^4}(\xi_4) b^*_-(\widetilde{\xi_4}) \overline{{}^\sharp u^{[J_1^3,J_2^3]}_{M_1^3 M_2^3})(\xi_3; m_3, j_3)} b^*_+(\xi_3; m_3, j_3)\big) \times \\
&\quad \overline{({}^\sharp u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2})(\xi_2; m_2, j_2)} a^*_+(\xi_2; m_2, j_2) \overline{{}^\sharp u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1})(\xi_1; m_1, j_1)} a^*_+(\xi_1; m_1, j_1)\big) \ .
\end{aligned}
\tag{5.24}
$$

and

$$
\begin{aligned}
\widetilde{V}_I^{(2)} = (2\pi)^{-3} &\sum_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} \overline{g^{(2)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4}} \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 \\
&\qquad \delta^3(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) \\
&\big({}^\sharp u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1}(\xi_1; m_1, j_1) a_+(\xi_1; m_1, j_1) {}^\sharp u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2}(\xi_2; m_2, j_2) a_+(\xi_2; m_2, j_2)\big) \times \\
&\qquad \big({}^\sharp u^{[J_1^3,J_2^3]}_{M_1^3 M_2^3}(\xi_3; m_3, j_3) b_+(\xi_3; m_3, j_3) \overline{{}^\sharp u^{[J_1^4,J_2^4]}_{M_1^4 M_2}}(\xi_4) b_-(\widetilde{\xi_4})\big) \ .
\end{aligned}
\tag{5.25}
$$

In the Fock space $\mathfrak{F}$ the interaction V_I is a highly singular operator due to the δ-distributions that occur in the $(V_I^{(*)})'s$ and the $(\widetilde{V}_I^{(*)})'s$ and because of the ultraviolet behavior of the functions $u^{[J_1^{(.)},J_2^{(.)}]}(.)$ involved.

In order to get well defined operators in $\mathfrak{F}$ we have to substitute smoother kernels $F^{(\alpha)}(\xi_1, \xi_2)$, $G^{(\alpha)}(\xi_3)$ and $\widetilde{G}^{(\alpha)}(\xi_4)$, where $\alpha = 1, 2$, for the δ-distributions.

We then obtain a new operator denoted by H_I and defined as follows in the Schrödinger representation.

$$
H_I = H_I^{(1)} + (H_I^{(1)})^* + H_I^{(2)} + (H_I^{(2)})^* \tag{5.26}
$$

Remark 5.1. *For the fermionic part of the interaction one could consider a kernel $G^{(\alpha)}(\xi_3, \xi_4)$ which is not split. Nevertheless this kernel should satisfy implicit conditions or should be very regular. It is better to consider a split kernel because the conditions that the split parts will have to satisfy will be more explicit and general.*

We have

$$
\begin{aligned}
&H_I^{(1)} = \\
&\quad (2\pi)^{-3} \sum_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} g^{(1)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 \\
&\qquad F^{(1)}(\xi_1, \xi_2) G^{(1)}(\xi_3) \widetilde{G}^{(1)}(\xi_4) \\
&\qquad \big(u^{[J_1^4,J_2^4]}_{M_1^4 M_2^4}(\xi_4) b^*_-(\widetilde{\xi_4}) \overline{{}^\sharp u^{[J_1^3,J_2^3]}_{M_1^3 M_2^3})(\xi_3; m_3, j_3)} b^*_+(\xi_3; m_3, j_3)\big) \times \\
&\quad \overline{({}^\sharp u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2})(\xi_2; m_2, j_2)} a^*_+(\xi_2; m_2, j_2) {}^\sharp u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1})(\xi_1; m_1, j_1) a_+(\xi_1; m_1, j_1)\big) \ .
\end{aligned}
\tag{5.27}
$$

$$(H_I^{(1)})^* = \\ (2\pi)^{-3} \sum_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} \overline{g^{(1)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4}} \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 \\ \overline{F^{(1)}(\xi_1,\xi_2) G^{(1)}(\xi_3) \widetilde{G}^{(1)}(\xi_4)} \\ \big(\overline{{}^\sharp u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1}}(\xi_1;m_1,j_1) a_+^*(\xi_1;m_1,j_1) {}^\sharp u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2}(\xi_2;m_2,j_2) a_+(\xi_2;m_2,j_2) \big) \times \\ \big({}^\sharp u^{[J_1^3,J_2^3]}_{M_1^3 M_2^3}(\xi_3;m_3,j_3) b_+(\xi_3;m_3,j_3) \overline{{}^\sharp u^{[J_1^4,J_2^4]}_{M_1^4 M_2}}(\xi_4) b_-(\widetilde{\xi_4}) \big) \ . \quad (5.28)$$

$$H_I^{(2)} = \\ (2\pi)^{-3} \sum_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} g^{(2)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 \\ F^{(2)}(\xi_1,\xi_2) G^{(2)}(\xi_3) \widetilde{G}^{(2)}(\xi_4) \\ \big(u^{[J_1^4,J_2^4]}_{M_1^4 M_2^4}(\xi_4) b_-^*(\widetilde{\xi_4}) \overline{{}^\sharp u^{[J_1^3,J_2^3]}_{M_1^3 M_2^3})}(\xi_3;m_3,j_3) b_+^*(\xi_3;m_3,j_3) \big) \times \\ \overline{({}^\sharp u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2})}(\xi_2;m_2,j_2) a_+^*(\xi_2;m_2,j_2) \overline{{}^\sharp u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1})}(\xi_1;m_1,j_1) a_+^*(\xi_1;m_1,j_1) \big) \ . \quad (5.29)$$

$$(H_I^{(2)})^* = (2\pi)^{-3} \sum_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} \overline{g^{(2)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4}} \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 \\ \overline{F^{(2)}(\xi_1,\xi_2) G^{(2)}(\xi_3) \widetilde{G}^{(2)}(\xi_4)} \\ \big({}^\sharp u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1}(\xi_1;m_1,j_1) a_+(\xi_1;m_1,j_1) {}^\sharp u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2}(\xi_2;m_2,j_2) a_+(\xi_2;m_2,j_2) \big) \times \\ \big({}^\sharp u^{[J_1^3,J_2^3]}_{M_1^3 M_2^3}(\xi_3;m_3,j_3) b_+(\xi_3;m_3,j_3) \overline{{}^\sharp u^{[J_1^4,J_2^4]}_{M_1^4 M_2}}(\xi_4) b_-(\widetilde{\xi_4}) \big) \ . \quad (5.30)$$

The total Hamiltonian is then

$$H = H_0 + H_I \quad (5.31)$$

We now give the conditions that the kernels $F^\alpha(.,.), G^{(\alpha)}(.), \widetilde{G}^{(\alpha)}(.)$ and the couplings constants $g^{(\alpha)}_{.}$ have to satisfy in order to associate with the formal operator H a well defined self-adjoint operator in $\mathfrak{F}$.

6. A self-adjoint Hamiltonian

Let $\mathfrak{D}$ denote the set of smooth vectors in $\mathfrak{F}$ for which only a finite number of components are different from zero and each component is smooth with a compact support. See [9] for a careful definition. H_0 is essentially self-adjoint on $\mathfrak{D}$. The spectrum of H_0 is $[0,\infty)$ and 0 is a simple eigenvalue with Ω as eigenvector.

The set of thresholds of H_0, denoted by $\mathcal{T}$, is given by

$$\mathcal{T} = \{p\, m_1 + q\, m_2 + r\, m_3; (p,\, q,\, r) \in \mathbb{N}^3 \text{ and } p+q+r \geq 1\} \ , \quad (6.1)$$

For each causal field concerning the massive particles we can choose either the canonical formalism or the helicity one. Nevertheless, from the physical point of

view, the helicity formalism is very important and from now on we restrict ourselves to this formalism for each particle. For any other choice of formalisms our results will the same because we can apply the same proof. Only constants and smallness conditions on the couplings constants would vary. We omit the details.

Thus, from now on, we omit the superscript H in the formulae.

We now need to estimate the functions $(u^{[J_1^i,J_2^i]}_{M_1^i M_2^i})(\xi_i;m_i,j_i)$,where $i=1,2,3$, and $u^{[J_1^4,J_2^4]}_{M_1^4 M_2^4}(\xi_4)$.

By (5.11) and one easily shows that there exist two constants C^i for $i=1,2,3$ such that

$$|(u^{[J_1^i,J_2^i]}_{M_1^i M_2^i})(\xi_i;m_i,j_i)| \leq C^i\big(1+|\mathbf{p}_i|\big)^{J_1^i+J_2^i-1/2} \tag{6.2}$$

Remark that C^i depends on J_1^i, J_2^i and j_i.

By (5.12) we obtain

$$|u^{[J_1^4,J_2^4]}_{M_1^4 M_2^4}(\xi_4)| \leq (2|\mathbf{p}_4|)^{J_1^4+J_2^4-1/2} \tag{6.3}$$

The estimate (6.3) is verified in the case of neutrinos and antineutrinos in the Standard Model.

From now on the kernels $F^{(\alpha)}(\xi_1,\xi_2)$,$G^{(\alpha)}(\xi_3)$ and $\widetilde{G}^{(\alpha)}(\xi_4)$ are supposed to satisfy the following hypothesis

Hypothesis 6.1. *For $\alpha=1,2$ we assume*

$$(i)\ \prod_{\beta=1,2,}\big(1+|\boldsymbol{p}_\beta|\big)^{J_1^\beta+J_2^\beta-1/2}F^{(\alpha)}(.,.)\in L^2(\Sigma_{j_1}\times\Sigma_{j_2})$$

$$(ii)\ \big(1+|\boldsymbol{p}_3|\big)^{J_1^3+J_2^3-1/2}G^{(\alpha)}(.)\in L^2(\Sigma_{j_3})$$

$$(iii)\ |\boldsymbol{p}_4|^{J_1^4+J_2^4-1/2}\widetilde{G}^{(\alpha)}(.)\in L^2(\mathbb{R}^3)$$

Set

$$_1F^{(1)}(\xi_1,\xi_2)=\overline{u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2}(\xi_2;m_2,j_2)}u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1})(\xi_1;m_1,j_1)F^{(1)}(\xi_1,\xi_2). \tag{6.4}$$

$$_2F^{(1)}(\xi_1,\xi_2)=u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2}(\xi_2;m_2,j_2)\overline{u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1})(\xi_1;m_1,j_1)F^{(1)}(\xi_1,\xi_2)}. \tag{6.5}$$

$$_1F^{(2)}(\xi_1,\xi_2)=\overline{u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2}(\xi_2;m_2,j_2)u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1})(\xi_1;m_1,j_1)}F^{(2)}(\xi_1,\xi_2). \tag{6.6}$$

$$_2F^{(2)}(\xi_1,\xi_2)=u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2}(\xi_2;m_2,j_2)u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1})(\xi_1;m_1,j_1)\overline{F^{(2)}(\xi_1,\xi_2)}. \tag{6.7}$$

For every $\Psi \in \mathfrak{D}$ we have

$$\begin{aligned}\| \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 ({}_1F^{(1)}(\xi_1,\xi_2)) a_+^*(\xi_2; m_2, j_2) a_+(\xi_1; m_1, j_1)\Psi\| \\ \leq C^1C^2 \| \big(\prod_{\beta=1,2} (1+|\mathbf{p}_\beta|)^{J_1^\beta+J_2^\beta-1/2}\big) F^{(1)}(.,.)\|_{L^2(\Sigma_{j_1}\times\Sigma_{j_2})} \\ \times \big((\frac{1}{m_1}+\frac{1}{m_2})\|H_0\Psi\| + \frac{1}{2}\|\Psi\| \ \big)\end{aligned} \tag{6.8}$$

$$\begin{aligned}\| \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 ({}_2F^{(1)}(\xi_1,\xi_2)) a_+^*(\xi_1; m_1, j_1) a_+(\xi_2; m_2, j_2)\Psi\| \\ \leq C^1C^2 \| \big(\prod_{\beta=1,2} (1+|\mathbf{p}_\beta|)^{J_1^\beta+J_2^\beta-1/2}\big) F^{(1)}(.,.)\|_{L^2(\Sigma_{j_1}\times\Sigma_{j_2})} \\ \times \big((\frac{1}{m_1}+\frac{1}{m_2})\|H_0\Psi\| + \frac{1}{2}\|\Psi\|\big)\end{aligned} \tag{6.9}$$

$$\begin{aligned}\| \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 ({}_1F^{(2)}(\xi_1,\xi_2)) a_+^*(\xi_2; m_2, j_2) a_+^*(\xi_1; m_1, j_1)\Psi\| \\ \leq C^1C^2 \| \big(\prod_{\beta=1,2} (1+|\mathbf{p}_\beta|)^{J_1^\beta+J_2^\beta-1/2}\big) F^{(2)}(.,.)\|_{L^2(\Sigma_{j_1}\times\Sigma_{j_2})} \\ \times \big((\frac{1}{m_1}+\frac{1}{m_2})\|H_0\Psi\| + \|\Psi\|\big)\end{aligned} \tag{6.10}$$

$$\begin{aligned}\| \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 ({}_2F^{(2)}(\xi_1,\xi_2)) a_+(\xi_1; m_1, j_1) a_+(\xi_2; m_2, j_2)\Psi\| \\ \leq C^1C^2 \| \big(\prod_{\beta=1,2} (1+|\mathbf{p}_\beta|)^{J_1^\beta+J_2^\beta-1/2}\big) F^{(2)}(.,.)\|_{L^2(\Sigma_{j_1}\times\Sigma_{j_2})} \\ \times (\frac{1}{m_1}+\frac{1}{m_2})\|H_0\Psi\|\end{aligned} \tag{6.11}$$

The estimates (6.8) – (6.11) are examples of N_τ estimates.The proof is similar to the one of [8, Proposition 3.7] and details are omitted.

Set

$$W(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) = \Big(|\mathbf{p}_4|^{J_1^4+J^4-1/2} \prod_{\beta=1,2,3} \big(1+|\mathbf{p}_\beta|\big)^{J_1^\beta+J_2^\beta-1/2}\Big) \tag{6.12}$$

$$C = (2\pi)^{-3}C^1C^2C^3 \big(\prod_{\beta=1}^{3} (1+2J_1^\beta)^2(1+2J_2^\beta)^2\big) 2^{J_1^4+J_2^4-1/2}(1+2J_1^4)(1+2J_2^4) \tag{6.13}$$

$$g = \sup_\alpha \sup_{M_1^1M_2^1M_1^2M_2^2M_1^3M_2^3M_1^4M_2^4} |g^{(\alpha)}_{M_1^1M_2^1M_1^2M_2^2M_1^3M_2^3M_1^4M_2^4}| \tag{6.14}$$

and

$$b = \frac{m_1m_2}{2(m_1+m_2)} \tag{6.15}$$

By (3.11), (4.7), (5.26) – (5.30) and (6.8) – (6.11) we finally get for every $\Psi \in \mathfrak{D}$

$$\|H_I\Psi\| \leq 2gC(\frac{1}{m_1} + \frac{1}{m_2})\times \sum_{\alpha=1,2} \|W(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)F^{(\alpha)}(.,.)G^{(\alpha)}(.)\widetilde{G}^{(\alpha)}(.)\|_{L^2(\Sigma_{j_1}\times\Sigma_{j_2}\times\Sigma_{j_3}\times\mathbb{R}^3)} \times (\|H_0\Psi\| + b\|\Psi\|) \ . \tag{6.16}$$

We then have the following theorem

Theorem 6.2. *Let $g_1 > 0$ be such that*

$$2Cg_1(\frac{1}{m_1} + \frac{1}{m_2}) \sum_{\alpha=1,2} \left(\int \left(W(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4)\right)^2 |F^{(\alpha)}(\xi_1, \xi_2)|^2 |G^{(\alpha)}(\xi_3)|^2 |\widetilde{G}^{(\alpha)}(\xi_4)|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \mathrm{d}\xi_4 \right)^{1/2} < 1 \ . \tag{6.17}$$

Then, for every g satisfying $g \leq g_1$, H is a self-adjoint operator in $\mathfrak{F}$ with domain $\mathcal{D}(H_0)$ and $\mathfrak{D}$ is a core for H.

By (6.16) the proof of the theorem follows from the Kato-Rellich theorem.

7. Main results

We now wish to give statements about the existence of an unique ground state for the Hamiltonian H together with the location of its spectrum and of its absolutely continuous spectrum. This is our first main result.

As in [9] and [3] our second main result is the proof that the spectrum of H is absolutely continuous in any interval $(\inf\sigma(H), \inf\sigma(H) + m_1 - \delta]$ for $\delta < m_1$ and for g sufficiently small whose smallness depends on δ.

We shall now make the following additional assumptions on the kernels $\widetilde{G}^{(\alpha)}(\xi_4)$.

Hypothesis 7.1. *There exist $K(\widetilde{G})$ and $\widetilde{K}(\widetilde{G})$ such that for $\alpha = 1, 2$, $i, l = 1, 2, 3$, and $\sigma \geq 0$,*

$$(i) \quad \left(|\boldsymbol{p}_4|^{J_1^4 + J_2^4 - 3/2}\right)\widetilde{G}^{(\alpha)}(\xi_4) \in L^2(\mathbb{R}^3) \ .$$

$$(ii) \quad \left(\int_{|\boldsymbol{p}_4| \leq \sigma} \left(|\boldsymbol{p}_4|^{2(J_1^4 + J_2^4) - 1} |\widetilde{G}^{(\alpha)}(\xi_4)|^2 \mathrm{d}\xi_4 \right) \right)^{1/2} \leq K(\widetilde{G})\sigma \ .$$

$$(iii-a) \quad \left(|\boldsymbol{p}_4|^{J_1^4 + J_2^4 - 1/2}\right)\left((\boldsymbol{p}_4 \cdot \nabla_{\boldsymbol{p}_4})\widetilde{G}^{(\alpha)}\right)(\xi_4) \in L^2(\mathbb{R}^3) \ .$$

$$(iii-b) \left(\int_{|\boldsymbol{p}_4| \leq \sigma} |\boldsymbol{p}_4|^{2(J_1^4 + J_2^4) - 1} |\left((\boldsymbol{p}_4 \cdot \nabla_{\boldsymbol{p}_4})\widetilde{G}^{(\alpha)}\right)(\xi_4)|^2 \mathrm{d}\xi_4 \right)^{1/2} \leq \widetilde{K}(\widetilde{G})\sigma \ .$$

$$(iii-c) \int_{\mathbb{R}^3} |\boldsymbol{p}_4|^{2(J_1^4 + J_2^4) - 1} (p_2^i)^2 (p_2^l)^2 \left| \frac{\partial^2 \widetilde{G}^{(\alpha)}}{\partial p_4^i \partial p_4^l}(\xi_4) \right|^2 \mathrm{d}\xi_4 < \infty \ .$$

The first main result is concerned with the existence of an unique ground state for H and with the location of the spectrum of H and of its absolutely continuous spectrum.

Theorem 7.2. *Assume that the kernels $F^{(\alpha)}$, $G^{(\alpha)}$ and $\widetilde{G}^{(\alpha)}$, $\alpha = 1, 2$, satisfy Hypothesis* (6.1) *and Hypothesis* (7.1(*i*))*. Then there exists $g_2 \in (0, g_1]$ such that H has an unique ground state for $g \le g_2$. Furthermore, setting*

$$E = \inf \sigma(H)$$

we have

$$\sigma(H) = \sigma_{\mathrm{ac}}(H) = [E, \infty)$$

with $E \le 0$.

In order to prove theorem 7.2 we first need to get an important result about the spectrum of the Hamiltonians with infrared cutoff.

Let us first define the cutoff operators which are the Hamiltonians with infrared cutoff with respect to the momentum of the massless fermion.

For that purpose, let $\chi_0(.) \in C^\infty(\mathbb{R}, [0,1])$ with $\chi_0 = 1$ on $(-\infty, 1]$ and $\chi_0 = 0$ on $[2, \infty]$. For $\sigma > 0$ and $\mathbf{p} \in \mathbb{R}^3$, we set

$$\begin{aligned} \chi_\sigma(\mathbf{p}) &= \chi_0(|\mathbf{p}|/\sigma) \ , \\ \tilde{\chi}^\sigma(\mathbf{p}) &= 1 - \chi_\sigma(\mathbf{p}) \ . \end{aligned} \tag{7.18}$$

The operator $H_{I,\sigma}$ is the interaction given by (5.26)-(5.30) associated with the kernel $\tilde{\chi}^\sigma(\mathbf{p}_4)\widetilde{G}^{(\alpha)}(\xi_4)$ instead of $\widetilde{G}^{(\alpha)}(\xi_4)$.

We then set

$$H_\sigma = H_0 + gH_{I,\sigma} \ . \tag{7.19}$$

We now introduce

$$\begin{aligned} &\Sigma_{4,\sigma} = \mathbb{R}^3 \cap \{|\mathbf{p}_4| < \sigma\} \ , \quad \Sigma_4^\sigma = \mathbb{R}^3 \cap \{|\mathbf{p}_4| \ge \sigma\} \\ &\mathfrak{F}_{4,\sigma} = \mathfrak{F}_a(L^2(\Sigma_{4,\sigma})) \ , \qquad \mathfrak{F}_4^\sigma = \mathfrak{F}_a(L^2(\Sigma_4^\sigma)) \ . \end{aligned} \tag{7.20}$$

$\mathfrak{F}_{4,\sigma} \otimes \mathfrak{F}^\sigma$ is the Fock space for the massless fermion.

Now, we set

$$\mathfrak{F}^\sigma = \mathfrak{F}_s^{[m_1,j_1]} \otimes \mathfrak{F}_s^{[m_2,j_2]} \otimes \mathfrak{F}_a^{[m_3,j_3]} \otimes \mathfrak{F}_4^\sigma \text{ and } \quad \mathfrak{F}_\sigma = \mathfrak{F}_{4,\sigma} \ . \tag{7.21}$$

and we have

$$\mathfrak{F} \simeq \mathfrak{F}^\sigma \otimes \mathfrak{F}_\sigma \ . \tag{7.22}$$

We further set

$$\begin{aligned} H_0^i &= \int w^i(\xi_i) a_+^*(\xi_i) a_+(\xi_i) \mathrm{d}\xi_i \ , \ \text{i=1,2,} \\ H_0^3 &= \int w^3(\xi_3) b_+^*(\xi_3) b_+(\xi_3) \mathrm{d}\xi_3 \ , \\ H_0^4 &= \int w^4(\xi_4) b_-^*(\widetilde{\xi_4}) b_-(\widetilde{\xi_4}) \mathrm{d}\xi_4 \ . \end{aligned} \tag{7.23}$$

and

$$\begin{aligned} H_0^{4,\sigma} &= \int_{|\mathbf{p}_4| > \sigma} w^4(\xi_4) b^*(\widetilde{\xi_4}) b(\widetilde{\xi_4}) \mathrm{d}\xi_4 \ , \\ H_{0,\sigma}^4 &= \int_{|\mathbf{p}_4| \le \sigma} w^4(\xi_4) b^*(\widetilde{\xi_4}) b(\widetilde{\xi_4}) \mathrm{d}\xi_4 \ . \end{aligned} \tag{7.24}$$

Then, on $\mathfrak{F}^\sigma \otimes \mathfrak{F}_\sigma$, we have

$$H_0^4 = H_0^{4,\sigma} \otimes \mathbf{1}_\sigma + \mathbf{1}^\sigma \otimes H_{0,\sigma}^4 \ , \tag{7.25}$$

where $\mathbf{1}^\sigma$ (resp. $\mathbf{1}_\sigma$) is the identity operator on $\mathfrak{F}^\sigma$ (resp. $\mathfrak{F}_\sigma$).

Using the definitions

$$H^\sigma = H_\sigma|_{\mathfrak{F}^\sigma} \quad \text{and} \quad H_0^\sigma = H_0|_{\mathfrak{F}^\sigma} \ ,$$

we get

$$H^\sigma = H_0^1 + H_0^2 + H_0^3 + H_0^{4,\sigma} + gH_{I,\sigma} \quad \text{on } \mathfrak{F}^\sigma \ , \tag{7.26}$$

and

$$H_\sigma = H^\sigma \otimes \mathbf{1}_\sigma + \mathbf{1}^\sigma \otimes H_{0,\sigma}^4 \quad \text{on } \mathfrak{F}^\sigma \otimes \mathfrak{F}_\sigma \ . \tag{7.27}$$

Now, for $\delta \in \mathbb{R}$ with $0 < \delta < m_3$, we define the sequence $(\sigma_n)_{n\geq 0}$ by

$$\begin{aligned} &\sigma_0 = 2m_3 + 1 \ , \\ &\sigma_1 = m_3 - \frac{\delta}{2} \ , \\ &\sigma_{n+1} = \gamma\sigma_n \text{ for } n \geq 1 \ , \end{aligned} \tag{7.28}$$

where

$$\gamma = 1 - \frac{\delta}{2m_3 - \delta} \ . \tag{7.29}$$

For $n \geq 0$, we then define the cutoff operators on $\mathfrak{F}^n = \mathfrak{F}^{\sigma_n}$ by

$$H^n = H^{\sigma_n}, \quad H_0^n = H_0^{\sigma_n}, \tag{7.30}$$

and we denote, for $n \geq 0$,

$$E^n = \inf \sigma(H^n). \tag{7.31}$$

We also define the cutoff operators on $\mathfrak{F}$ by

$$H_n = H_{\sigma_n}, \quad H_{0,n} = H_{0,\sigma_n}, \tag{7.32}$$

and we denote, for $n \geq 0$,

$$E_n = \inf \sigma(H_n). \tag{7.33}$$

Note that

$$E^n = E_n \tag{7.34}$$

One easily shows that, for $g \leq g_1$,

$$|E^n| = |E_n| \leq gbK(F, G, \widetilde{G}) \frac{1}{1 - g_1 K(F, G, \widetilde{G})} \tag{7.35}$$

See [9, 3] for a proof.

We now set

$$\begin{aligned} K(F, G, \widetilde{G}) = 2C(\frac{1}{m_1} + \frac{1}{m_2}) \times \\ \sum_{\alpha=1,2} \|W(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) F^{(\alpha)}(.,.) G^{(\alpha)}(.) \widetilde{G}^{(\alpha)}(.)\|_{L^2(\Sigma_{j_1} \times \Sigma_{j_2} \times \Sigma_{j_3} \times \mathbb{R}^3)} \ . \end{aligned} \tag{7.36}$$

$$\begin{aligned} K_1(F, G, \widetilde{G}) &= \frac{1}{1 - g_1 K(F, G, \widetilde{G})} \\ K_2(F, G, \widetilde{G}) &= \frac{b}{\left(1 - g_1 K(F, G, \widetilde{G})\right)^2} \end{aligned} \tag{7.37}$$

$$\widetilde{K}(F,G) = 2C(\frac{1}{m_1}+\frac{1}{m_2})\times$$
$$\sum_{\alpha=1,2}\|\big(\prod_{\beta=1,2,3}(1+|\mathbf{p}_\beta|)^{J_1^\beta+J_2^\beta-1/2}\big)F^{(\alpha)}(.,.)G^{(\alpha)}(.)\|_{L^2(\Sigma_{j_1}\times\Sigma_{j_2}\times\Sigma_{j_3})}\ . \tag{7.38}$$

$$\widetilde{D}_\delta(F,G,\widetilde{G}) = \max\left\{\frac{4(2m_3+1)\gamma}{2m_3-\delta},\ 2\right\}\times$$
$$\widetilde{K}(F,G)\big(2m_3K_1(F,G,\widetilde{G})+K_2(F,G,\widetilde{G})\big) \tag{7.39}$$

Let $g_\delta^{(1)}$ be such that

$$0 < g_\delta^{(1)} < \min\left\{1,\ g_1,\ \frac{\gamma-\gamma^2}{3\widetilde{D}_\delta(F,G,\widetilde{G})}\right\}\ . \tag{7.40}$$

and let

$$g_3 = \frac{1}{4K(F,G,\widetilde{G})} \tag{7.41}$$

Setting

$$g_\delta^{(2)} = \inf\{g_3, g_\delta^{(1)}\} \tag{7.42}$$

and applying the same method as the one used for proving proposition 4.1 in [3] we finally get the following result

Proposition 7.3. *Suppose that the kernels $F^{(\alpha)}(.,.)$, $G^{(\alpha)}(.)$ and $\widetilde{G}^{(\alpha)}(.)$, $\alpha = 1,2$, satisfy Hypothesis 6.1 and 7.1(ii) . Then, for $g \le g_\delta^{(2)}$, E^n is a simple eigenvalue of H^n for $n \ge 1$, and H^n does not have spectrum in the interval $(E^n,\ E^n + (1-3g\frac{\widetilde{D}_\delta(F,G,\widetilde{G})}{\gamma})\sigma_n)$.*

7.0.1. ***Proof of theorem 7.2.***

We adapt the proof of theorem 3.3 in [9]. By Proposition 7.3 H^n has an unique ground state, denoted by ϕ^n, in $\mathfrak{F}^n$ such that

$$H^n\phi^n = E^n\phi^n,\quad \phi^n\in\mathcal{D}(H^n),\quad \|\phi^n\| = 1,\quad n\ge 1\ . \tag{7.43}$$

Therefore H_n has an unique normalized ground state in $\mathfrak{F}$, given by $\tilde{\phi}_n = \phi^n\otimes\Omega_n$, where Ω_n is the vacuum state in $\mathfrak{F}_n$,

$$H_n\tilde{\phi}_n = E^n\tilde{\phi}_n,\quad \tilde{\phi}_n\in\mathcal{D}(H_n),\quad \|\tilde{\phi}_n\| = 1,\quad n\ge 1\ .$$

Let $H_{I,n}$ be the interaction H_{I,σ_n}. It follows from the pull-through formula that

$$(H_0+H_{I,n})b_-(\widetilde{\xi_4})\tilde{\phi}_n = E_nb_-(\widetilde{\xi_4})\tilde{\phi}_n - w^4(\xi_4)b_-(\widetilde{\xi_4})\tilde{\phi}_n - \big(V_n^1(\xi_4)+V_n^2(\xi_4)\big)\tilde{\phi}_n \tag{7.44}$$

where

$$V_n^1(\xi_4) = (2\pi)^{-3} \sum_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} g^{(1)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$
$$F^{(1)}(\xi_1,\xi_2) G^{(1)}(\xi_3) \tilde{\chi}^{\sigma_n}(\mathbf{p}_4) \widetilde{G}^{(1)}(\xi_4)$$
$$\big(u^{[J_1^4,J_2^4]}_{M_1^4 M_2^4}(\xi_4) \overline{u^{[J_1^3,J_2^3]}_{M_1^3 M_2^3}})(\xi_3; m_3, j_3) b_+^*(\xi_3; m_3, j_3)\big)$$
$$\big(\overline{u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2}})(\xi_2; m_2, j_2) a_+^*(\xi_2; m_2, j_2) u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1})(\xi_1; m_1, j_1) a_+(\xi_1; m_1, j_1)\big) . \tag{7.45}$$

and

$$V_n^2(\xi_4) = (2\pi)^{-3} \sum_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} g^{(2)}_{M_1^1 M_2^1 M_1^2 M_2^2 M_1^3 M_2^3 M_1^4 M_2^4} \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$
$$F^{(2)}(\xi_1,\xi_2) G^{(2)}(\xi_3) \tilde{\chi}^{\sigma_n}(\mathbf{p}_4) \widetilde{G}^{(2)}(\xi_4)$$
$$\big(u^{[J_1^4,J_2^4]}_{M_1^4 M_2^4}(\xi_4) \overline{u^{[J_1^3,J_2^3]}_{M_1^3 M_2^3}})(\xi_3; m_3, j_3) b_+^*(\xi_3; m_3, j_3)\big)$$
$$\big(\overline{u^{[J_1^2,J_2^2]}_{M_1^2 M_2^2}})(\xi_2; m_2, j_2) a_+^*(\xi_2; m_2, j_2) \overline{u^{[J_1^1,J_2^1]}_{M_1^1 M_2^1}})(\xi_1; m_1, j_1) a_+^*(\xi_1; m_1, j_1)\big) . \tag{7.46}$$

We obtain

$$\|\big(V_n^1(\xi_4) + V_n^2(\xi_4)\big)\tilde{\phi}_n\| \leq gC|\mathbf{p}_4|^{J_1^4+J_2^4-1/2}$$
$$\sum_{\alpha=1,2} |\widetilde{G}^{(\alpha)}(\xi_4)| \big(\| \prod_{\beta=1,2,3} (1+|\mathbf{p}_\beta|)^{J_1^\beta+J_2^\beta-1/2} F^{(\alpha)}(.,.) G^{(\alpha)}(.)\|_{L^2(\Sigma_{j_1}\times\Sigma_{j_2}\times\Sigma_{j_3})}\big)$$
$$\big((\frac{1}{m_1}+\frac{1}{m_2})\|H_0\tilde{\phi}_n\| + 3/2\big) \tag{7.47}$$

It follows from (6.16) and (7.35) that, for every $g \leq g_1$,

$$\|H_{I,n}\tilde{\phi}_n\| \leq gK(F,G,\widetilde{G})(\|H_0\tilde{\phi}_n\| + b) \tag{7.48}$$

This yields

$$\|H_0\tilde{\phi}_n\| \leq |E_n| + gK(F,G,\widetilde{G})(\|H_0\tilde{\phi}_n\| + b) \tag{7.49}$$

By (7.35) and (7.37) we get

$$\|H_0\tilde{\phi}_n\| \leq gbK(F,G,\widetilde{G})\big(K_2(F,G,\widetilde{G}) + K_1(F,G,\widetilde{G})\big) \tag{7.50}$$

uniformly with respect to n.

Setting $M = g_1 bK(F,G,\widetilde{G})\big(K_2(F,G,\widetilde{G}) + K_1(F,G,\widetilde{G})\big)$ we obtain

$$\|b_-(\widetilde{\xi_4})\tilde{\phi}_n\| \leq gC|\mathbf{p}_4|^{J_1^4+J_2^4-3/2}$$
$$\sum_{\alpha=1,2} |\widetilde{G}^{(\alpha)}(\xi_4)| \big(\| \prod_{\beta=1,2,3} (1+|\mathbf{p}_\beta|)^{J_1^\beta+J_2^\beta-1/2} F^{(\alpha)}(.,.) G^{(\alpha)}(.)\|_{L^2(\Sigma_{j_1}\times\Sigma_{j_2}\times\Sigma_{j_3})}\big)$$
$$\big((\frac{1}{m_1}+\frac{1}{m_2})M + 3/2\big) \tag{7.51}$$

Thus by Hypothesis 6.1 and 7.1(i) and from (7.51) there exists a constant $\mathcal{O}(F,G,\widetilde{G})>0$ such that

$$\int \|b_-(\widetilde{\xi_4})\tilde{\phi}_n\|^2 \mathrm{d}\xi_4 \leq g^2\mathcal{O}(F,G,\widetilde{G}) \tag{7.52}$$

uniformly with respect to n.

Since $\|\tilde{\phi}_n\|=1$, there exists a subsequence $(n_k)_{k\geq 1}$, converging to ∞ such that $(\tilde{\phi}_{n_k})_{k\geq 1}$ converges weakly to a state $\tilde{\phi}\in\mathfrak{F}$. By adapting the proof of theorem 4.1 in [8, 1] it follows from (7.52) that $\tilde{\phi}\neq 0$ for g sufficiently small, i.e.,$g\leq g_2\leq g_\delta^{(2)}$ and is a ground state of H.

By using the method developed in [32] (see [32, corollary 3.4]) we prove that this normalized ground state is unique(up to a phase).

The result about $\sigma(H)$ is the consequence of the existence of a ground state for H and of the existence of asymptotic Fock representations for the CCR and CAR associated with the $a_+^\sharp(\xi_i)'s$, $i=1,2$, the $b_+^\sharp(\xi_3)'s$ and the $b_-^\sharp(\widetilde{\xi_4})'s$.

For $f^i\in C_0^\infty(\Sigma_{j_i})$, $i=1,2,3$, and for $f^4\in C_0^\infty(\mathbb{R}^3)$ one defines on $\mathcal{D}(H_0)$ the operators

$$a_+^{\sharp,t}(f_i)=\mathrm{e}^{itH}\mathrm{e}^{-itH_0}a_+^\sharp(f_i)\mathrm{e}^{itH_0}\mathrm{e}^{itH} \tag{7.53}$$
$$b_+^{\sharp,t}(f_3)=\mathrm{e}^{itH}\mathrm{e}^{-itH_0}b_+^\sharp(f_3)\mathrm{e}^{itH_0}\mathrm{e}^{itH} \tag{7.54}$$
$$b_-^{\sharp,t}(f_4)=\mathrm{e}^{itH}\mathrm{e}^{-itH_0}b_-^\sharp(f_4)\mathrm{e}^{itH_0}\mathrm{e}^{itH}\ . \tag{7.55}$$

By adapting the proof given in [31] we prove that the strong limits of $a_+^{\sharp,t}(f_i)$, $i=1,2$, $b_+^{\sharp,t}(f_3)$ and $b_-^{\sharp,t}(f_4)$ when $t\to\pm\infty$ exist for every $\psi\in\mathcal{D}(H_0)$:

$$\lim_{t\to\pm\infty} a_+^{\sharp,t}(f_i):=a_+^{\sharp,\pm}(f_i) \tag{7.56}$$
$$\lim_{t\to\pm\infty} b_+^{\sharp,t}(f_3):=b_+^{\sharp,\pm}(f_3) \tag{7.57}$$
$$\lim_{t\to\pm\infty} b_-^{\sharp,t}(f_4):=b_-^{\sharp,\pm}(f_4)\ . \tag{7.58}$$

The operators $a_+^{\sharp,\pm}(f_i)$, $i=1,2$, satisfy the CCR and the operators $b_+^{\sharp,\pm}(f_3)$ and $b_-^{\sharp,\pm}(f_4)$ satisfy the CAR and we have

$$a_+^{\sharp,\pm}(f_i)\widetilde{\phi}=0 \tag{7.59}$$
$$b_+^{\sharp,\pm}(f_3)\widetilde{\phi}=0 \tag{7.60}$$
$$b_-^{\sharp,\pm}(f_4)\widetilde{\phi}=0\ . \tag{7.61}$$

where $\widetilde{\phi}$ is the normalized ground state of H.

By adapting [31] it follows from (7.56) – (7.61) that the absolutely continuous spectrum of H is $[\inf\sigma(H),\infty)$. We omit the details.

This concludes the proof of theorem 7.2.

Our second main result is devoted to the study of spectrum above the energy of the ground state.

Let p be the operator in $L^2(\mathbb{R}^3)$ associated to the position of the neutrinos and antineutrinos:

$$p = i\nabla_{\mathbf{p}_2} \ ,$$

and set

$$\langle p\rangle = (1+|p|^2)^{1/2}$$

The second quantized version $\mathrm{d}\Gamma(\langle p\rangle)$ is a self-adjoint operator in $\mathfrak{F}_a(L^2(\mathbb{R}^3)$. We then define the position operator P for the neutrinos and antineutrinos in $\mathfrak{F}$ by

$$P = \mathbf{1}\otimes\mathbf{1}\otimes\mathrm{d}\Gamma(\langle p\rangle)\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathrm{d}\Gamma(\langle p\rangle)\otimes\mathbf{1}\otimes\mathbf{1} \ . \tag{7.62}$$

We then have our second main result devoted to the spectrum above the energy of the ground state and below the first threshold.

Theorem 7.4. *Suppose that the kernels $F^{(\alpha)}(.,.)$, $G^{(\alpha)}(.)$ and $\widetilde{G}^{(\alpha)}(.)$, $\alpha=1,2$, satisfy Hypothesis 6.1 and 7.1. For any $\delta>0$ satisfying $0<\delta<m_3$ there exists $g_\delta>0$ for $0<g\le g_\delta$:*

(i) *The spectrum of H in $(E, E+m_3-\delta]$ is absolutely continuous.*

(ii) *For $s>1/2$, φ and $\psi\in\mathfrak{F}$ the limits*

$$\lim_{\epsilon\to 0}\big(\varphi,\ \langle P\rangle^{-s}(H-\lambda\pm i\epsilon)^{-1}\langle P\rangle^{-s}\psi\big)$$

exist uniformly for λ in every compact subset of $(E, E+m_3-\delta]$.

(iii) *For $s\in(1/2,1)$ the map*

$$\lambda\to\langle P\rangle^{-s}(H-\lambda\pm i0)^{-1}\langle P\rangle^{-s}$$

is locally Hölder continuous of degree $s-1/2$ in $(E, E+m_1-\delta]$.

(iv) *For $s\in(1/2,1)$ and $f\in C_0^\infty\big((E,E+m_1-\delta)\big)$ we have*

$$\|(P+1)^{-s}\mathrm{e}^{-itH}f(H)(P+1)^{-s}\| = \mathcal{O}\big(t^{-(s-1/2)}\big)\,.$$

7.0.2. ***Proof of theorem 7.4.***

Proof. .The following proposition will be fundamental for the proof of theorem 7.4. A straightforward but lengthy computation shows the following fundamental estimates

Proposition 7.5. *There exists $C(J_1^4, J_2^4)>0$ such that we have*

$$|p_4^i|\left|\frac{\partial u^{[J_1^4,J_2^4]}_{M_1^4M_2^4}}{\partial p_4^i}(\xi_4)\right| \le C(J_1^4,J_2^4)|\boldsymbol{p}_4|^{J_1^4+J_2^4-1/2}$$

$$|p_4^i||p_4^l|\left|\frac{\partial^2 u^{[J_1^4,J_2^4]}_{M_1^4M_2^4}}{\partial p_4^i\partial p_4^l}(\xi_4)\right| \le C(J_1^4,J_2^4)|\boldsymbol{p}_4|^{J_1^4+J_2^4-1/2}$$

for $i,l=1,2,3$.

In the proof of proposition 7.5 we explicitly use the norm of the operators $\mathcal{J}^{(2,.)}_{.}$ associated with the l^2-norm of $\mathbb{C}^{(2J^2+1)}$.

We now introduce a strict Mourre inequality.

Let us set

$$\tau = 1-\frac{\delta}{2(2m_3-\delta)} \ . \tag{7.63}$$

We now introduce $\chi^{(\tau)} \in C^\infty(\mathbb{R}, [0,1])$ be such that

$$\chi^{(\tau)}(\lambda) = \begin{cases} 1 & \text{for } \lambda \in (-\infty, \tau] , \\ 0 & \text{for } \lambda \in [1, \infty) . \end{cases} \tag{7.64}$$

and we set, for all $\mathbf{p}_4 \in \mathbb{R}^3$ and $n \geq 1$,

$$\chi_n^{(\tau)}(\mathbf{p}_4) = \chi^{(\tau)}\left(\frac{|\mathbf{p}_4|}{\sigma_n}\right), \tag{7.65}$$

$$a_n^{(\tau)} = \chi_n^{(\tau)}(\mathbf{p}_4)\frac{1}{2}\left(\mathbf{p}_4 \cdot i\nabla_{\mathbf{p}_4} + i\nabla_{\mathbf{p}_4} \cdot \mathbf{p}_4\right)\chi_n^{(\tau)}(\mathbf{p}_4) , \tag{7.66}$$

and

$$A_n^{(\tau)} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathrm{d}\Gamma(a_n^{(\tau)}) , \tag{7.67}$$

The operators $a_n^{(\tau)}$ and $A_n^{(\tau)}$ are self-adjoint and we also have

$$a_n^{(\tau)} = \frac{1}{2}\left(\chi_n^{(\tau)}(\mathbf{p}_4)^2\mathbf{p}_4 \cdot i\nabla_{\mathbf{p}_4} + i\nabla_{\mathbf{p}_4} \cdot \mathbf{p}_4\,\chi_n^{(\tau)}(\mathbf{p}_4)^2\right) . \tag{7.68}$$

Let now N be the smallest integer such that

$$N\gamma \geq 1 . \tag{7.69}$$

Let us define

$$\epsilon_\gamma = \min\left\{\frac{1}{2N}\Big(1 - \frac{3g\widetilde{D}_\delta(F,G,\widetilde{G})}{\gamma} - \gamma\Big), \frac{\tau-\gamma}{4}\right\} , \tag{7.70}$$

and choose $f \in C_0^\infty(\mathbb{R})$ such that $0 \leq f \leq 1$ and

$$f(\lambda) = \begin{cases} 1 & \text{if } \lambda \in [(\gamma-\epsilon_\gamma)^2, \gamma+\epsilon_\gamma] , \\ 0 & \text{if } \lambda > \gamma + 2\epsilon_\gamma , \\ 0 & \text{if } \lambda < (\gamma - 2\epsilon_\gamma)^2 . \end{cases} \tag{7.71}$$

We now define, for $n \geq 1$,

$$f_n(\lambda) = f\left(\frac{\lambda}{\sigma_n}\right) , \tag{7.72}$$

Let P^n denote the ground state projection of H^n and let $H_{0,n}^4$ denote H_{0,σ_n}^4.

It follows from Proposition 7.3 that, for $n \geq 1$ and $g \leq g_\delta^{(2)}$,

$$f_n(H_n - E_n) = P^n \otimes f_n(H_{0,n}^4) . \tag{7.73}$$

For $E = \inf\sigma(H)$ and any interval Δ, let $E_\Delta(H-E)$ be the spectral projection for the operator $(H-E)$ onto Δ. Consider, for $n \geq 1$,

$$\Delta_n = [(\gamma-\epsilon_\gamma)^2\sigma_n, (\gamma+\epsilon_\gamma)\sigma_n] . \tag{7.74}$$

Now, by adapting the proof of theorem 5.1 (Mourre inequality) in [3] and by applying proposition 7.5 together with Hypothesis 6.1 and Hypothesis 7.1(ii), (iii-a) and (iii-b), we prove the existence of a constant $\widetilde{C}_\delta(F,G,\widetilde{G}) > 0$ such that for every $g \leq \inf\left(g_2, g_\delta^{(2)}\right)$ we have

$$f_n(H-E)[H, iA_n^{(\tau)}]f_n(H-E) \geq \frac{\gamma^2}{N^2}\sigma_n f_n(H-E)^2 - g\sigma_n\tilde{C}_\delta(F,G,\widetilde{G}) . \tag{7.75}$$

Multiplying both sides of (7.75) with $E_{\Delta_n}(H-E)$ we obtain

$$E_{\Delta_n}(H-E)[H,\, iA_n^{(\tau)}]E_{\Delta_n}(H-E) \geq \left(\frac{\gamma^2}{N^2} - g\tilde{C}_\delta(F,G,\widetilde{G})\right)\sigma_n E_{\Delta_n}(H-E)\,. \tag{7.76}$$

Choosing a constant $g_\delta^{(3)}$ such that

$$g_\delta^{(3)} < \min\left\{g_2,\, g_\delta^{(2)}, \frac{\gamma^2}{N^2}\frac{1}{\tilde{C}_\delta(F,G,\widetilde{G})}\right\}, \tag{7.77}$$

we finally get the following strict Mourre inequality for every $g \leq g_\delta^{(3)}$ and for $n \geq 1$

$$E_{\Delta_n}(H-E)\,[H,\, iA_n^{(\tau)}]\,E_{\Delta_n}(H-E) \geq C_\delta(F,G,\widetilde{G})\frac{\gamma^2}{N^2}\,\sigma_n\, E_{\Delta_n}(H-E)\,. \tag{7.78}$$

where

$$C_\delta(F,G,\widetilde{G}) = (1 - N^2\tilde{C}_\delta(F,G,\widetilde{G})g_\delta^{(3)}/\gamma^2) > 0\ . \tag{7.79}$$

After proving a strict Mourre inequality it remains to prove that H is of class $C^2\big(A_n^{(\tau)}\big)$ in order to apply the commutator theory. See [35, 2, 41, 21, 25, 23].

In fact, according to [41], it suffices to prove that H is locally of class $C^2\big(A_n^{(\tau)}\big)$ in $(-\infty, m_3 - \delta/2)$.

This means that, for any $\varphi \in C_0^\infty(-\infty, m_3-\delta/2)$, $\varphi(H)$ is of class $C^2\big(A_n^{(\tau)}\big)$, i.e., $t \to \mathrm{e}^{-iA_n^{(\tau)}t}\varphi(H)\mathrm{e}^{iA_n^{(\tau)}t}\psi$ is twice continuously differentiable for all $\varphi \in C_0^\infty(-\infty, m_3-\delta/2)$ and $\psi \in \mathfrak{F}$.

Set

$$A_{n,t}^{(\tau)} = \frac{\mathrm{e}^{-iA_n^{(\tau)}t} - 1}{t} \tag{7.80}$$

By using the proof given in [9] H is locally of class $C^2\big(A_n^{(\tau)}\big)$ in $(-\infty, m_3 - \delta/2)$ if we show that

$$\sup_{0<|t|\leq 1} \|,[A_{n,t}^{(\tau)},\, [A_{n,t}^{(\tau)},\, H]](H+i)^{-1}\| < \infty \tag{7.81}$$

The operator $a_n^{(\tau)}$ is associated to the following C^∞- vector field in $\mathbb{R}^3$:

$$V_n^{(\tau)}(\mathbf{p}_4) = \chi_n^{(\tau)}(\mathbf{p}_4)^2\mathbf{p}_4 \tag{7.82}$$

Let $\phi_{n,t}^{(\tau)}(.) : \mathbb{R}^3 \to \mathbb{R}^3$ be the corresponding flow generated by $V_n^{(\tau)}(\mathbf{p}_4)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi_{n,t}^{(\tau)}(\mathbf{p}_4) &= V_n^{(\tau)}\big(\phi_{n,t}(\mathbf{p}_4)\big) \\ \phi_{n,0}^{(\tau)}(\mathbf{p}_4) &= \mathbf{p}_4 \end{aligned} \tag{7.83}$$

We have

$$\mathrm{e}^{-|t|}|\mathbf{p}_4| \leq |\phi_{n,t}^{(\tau)}(\mathbf{p}_4)| \leq \mathrm{e}^{|t|}|\mathbf{p}_4| \tag{7.84}$$

$\phi_{n,t}^{(\tau)}(\mathbf{p}_4)$ induces a one-parameter group of unitary operators $U_n^{(\tau)}(t)$ in $L^2(\mathbb{R}^3)$ defined by

$$\left(U_n^{(\tau)}(t)f\right)(\mathbf{p}_4) = \left(\det \nabla\phi_{n,t}^{(\tau)}(\mathbf{p}_4)\right)^{\frac{1}{2}} f(\phi_{n,t}^{(\tau)}(\mathbf{p}_4)). \tag{7.85}$$

$a_n^{(\tau)}$ is the generator of $U_n^{(\tau)}(t)$, i.e.,

$$U_n^{(\tau)}(t) = \mathrm{e}^{-ia_n^{(\tau)}t} \tag{7.86}$$

We have,for every $\psi \in \mathcal{D}(H)$

$$[A_{n,t}^{(\tau)},\, [A_{n,t}^{(\tau)},\, H]]\psi = \frac{1}{t^2}\mathrm{e}^{2iA_n^{(\tau)}t}\left(\mathrm{e}^{-2iA_n^{(\tau)}t}H\mathrm{e}^{2iA_n^{(\tau)}t} - 2\mathrm{e}^{-iA_n^{(\tau)}t}H\mathrm{e}^{iA_n^{(\tau)}t} + H\right)\psi \tag{7.87}$$

In particular we get

$$[A_{n,t}^{(\tau)},\, [A_{n,t}^{(\tau)},\, H_0]]\psi = \frac{1}{t^2}\mathrm{e}^{2iA_n^{(\tau)}t}\left(\mathrm{d}\Gamma(|\phi_{n,2t}^{(\tau)}(\mathbf{p}_4)| - 2|\phi_{n,t}^{(\tau)}(\mathbf{p}_4)| + |\mathbf{p}_4|)\right)\psi\ . \tag{7.88}$$

We note that

$$\frac{1}{t^2}\left||\phi_{n,2t}^{(\tau)}(\mathbf{p}_4)| - 2|\phi_{n,t}^{(\tau)}(\mathbf{p}_4)| + |\mathbf{p}_4|\right| \leq \sup_{|s|\leq 2|t|}\left|\frac{\partial^2}{\partial s^2}|\phi_{n,s}(\mathbf{p}_4)|\right|\ , \tag{7.89}$$

There exists a constant $c_n > 0$ such that

$$\left|\frac{\partial^2}{\partial s^2}|\phi_{n,s}(\mathbf{p}_4)|\right| \leq c_n|\phi_{n,t}^{(\tau)}(\mathbf{p}_4)| \leq c_n|\mathbf{p}_4| \tag{7.90}$$

This yields

$$\sup_{0\leq|t|\leq 1}\|[A_{n,t}^{(\tau)},\, [A_{n,t}^{(\tau)},\, H_0]]\,(H_0+1)^{-1}\,\| \leq c_n\mathrm{e}^2 \tag{7.91}$$

Let

$$\mathcal{G}^{(\alpha)}(\mathbf{p}_4) = u_{M_1^4 M_2^4}^{[J_1^4,J_2^4]}(\xi_4)\widetilde{G}^{(\alpha)}(\xi_4) \tag{7.92}$$

and

$$\mathcal{G}_t^{(\alpha)}(\mathbf{p}_4) = \left(\mathrm{e}^{-ia_n^{(\tau)}t}\mathcal{G}^{(\alpha)}\right)(\mathbf{p}_4) \tag{7.93}$$

It follows from (5.26) – (5.30) that we can write

$$H_I = \sum_{\alpha=1,2} H_I(F^{(\alpha)}, G^{(\alpha)}, \mathcal{G}^{(\alpha)}) \tag{7.94}$$

We then have, for every $\psi \in \mathcal{D}(H)$

$$[A_{n,t}^{(\tau)},\, [A_{n,t}^{(\tau)},\, H_I]]\psi = \sum_{\alpha=1,2}\frac{1}{t^2}\mathrm{e}^{2iA_n^{(\tau)}t}$$
$$\left(H_I(F^{(\alpha)}, G^{(\alpha)}, \mathcal{G}_{2t}^{(\alpha)}) - 2H_I(F^{(\alpha)}, G^{(\alpha)}, \mathcal{G}_t^{(\alpha)}) + H_I(F^{(\alpha)}, G^{(\alpha)}, \mathcal{G}^{(\alpha)})\right)\psi \tag{7.95}$$

By (6.16),(7.36) and (7.38) we get

$$\|[A_{n,t}^{(\tau)},\, [A_{n,t}^{(\tau)},\, H_I]]\psi\| \leq g\widetilde{K}(F,G)$$
$$\left(\frac{1}{t^2}\sum_{\alpha=1,2}\|\mathcal{G}_{2t}^{(\alpha)}(.) - 2\mathcal{G}_t^{(\alpha)}(.) + \mathcal{G}^{(\alpha)}(.)\|_{L^2(\mathbb{R}^3)}\right)^{1/2}(\|H_0\psi\| + b\|\psi\|)\,. \tag{7.96}$$

Note that, for $0 \leq |t| \leq 1$,

$$\left(\frac{1}{t^2}\sum_{\alpha=1,2}\|\mathcal{G}^{(\alpha)}_{2t}(.)-2\mathcal{G}^{(\alpha)}_{t}(.)+\mathcal{G}^{(\alpha)}(.)\|_{L^2(\mathbb{R}^3)}\right)^{1/2}$$
$$\leq \sup_{0<|s|\leq 2}\Big(\sum_{\alpha=1,2}\left\|\frac{\partial^2}{\partial s^2}\mathcal{G}^{(\alpha)}_s(.)\right\|^2_{L^2(\mathbb{R}^3)}\Big)^{1/2}. \quad (7.97)$$

with

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial s^2}(\mathrm{e}^{-ia^{(\tau)}_n s}\mathcal{G}^{(\alpha)})\right)(\mathbf{p}_4)\\
&=\frac{1}{4}\left(\mathrm{e}^{-ia^{(\tau)}_n s}((\mathrm{div}V^{(\tau)}_n(\mathbf{p}_4))^2\mathcal{G}^{(\alpha)})\right)(\mathbf{p}_4)\\
&+\frac{1}{2}\left(\mathrm{e}^{-ia^{(\tau)}_n s}((\mathrm{div}V^{(\tau)}_n(\mathbf{p}_4))V^{(\tau)}_n(\mathbf{p}_4)\cdot\nabla_{\mathbf{p}_4}\mathcal{G}^{(\alpha)})\right)(\mathbf{p}_4)\\
&+\frac{1}{2}\left(\mathrm{e}^{-ia^{(\tau)}_n s}(\sum_{i,j=1}^{3}(V^{(\tau),i}_n(\mathbf{p}_4)(\frac{\partial^2}{\partial p^i_4\partial p^j_4}V^{(\tau),j}_n(\mathbf{p}_4)))\mathcal{G}^{(\alpha)})\right)(\mathbf{p}_4) \qquad (7.98)\\
&+\frac{1}{2}\left(\mathrm{e}^{-ia^{(\tau)}_n s}(\sum_{i,j=1}^{3}V^{(\tau),i}_n(\mathbf{p}_4)\frac{\partial V^{(\tau),j}_n}{\partial p^i_4}(\mathbf{p}_4)\frac{\partial}{\partial p^j_4}\mathcal{G}^{(\alpha)})\right)(\mathbf{p}_4)\\
&+\frac{1}{2}\left(\mathrm{e}^{-a^{(\tau)}_n s}(\sum_{i,j=1}^{3}V^{(\tau),i}_n(\mathbf{p}_4)V^{(\tau),j}_n(\mathbf{p}_4)\frac{\partial^2}{\partial p^i_4\partial p^j_4}\mathcal{G}^{(\alpha)})\right)(\mathbf{p}_4)\ .
\end{aligned}$$

Combining the properties of the C^∞ field $V^{(\tau)}_n(\mathbf{p}_4)$ with Hypothesis 6.1 and 7.1 together with Proposition 7.5 and by mimicking the proof of theorem 5.1 in [9] we finally prove (7.81). It follows H is locally of class $C^2\big(A^{(\tau)}_n\big)$ in $(-\infty, m_3-\delta/2)$.

By applying the commutator theory (see [35, 2, 41, 21, 25, 23]), we then get the following Limiting Absorption Principle

Theorem 7.6. *Suppose that the kernels $F^{(\alpha)}(.,.)$,$G^{(\alpha)}(.)$ and $\widetilde{G}^{(\alpha)}(.)$, $\alpha=1,2$, satisfy Hypothesis 6.1 and Hypothesis 7.1. Then, for any $\delta>0$ satisfying $0<\delta<m_3$, there exists $g_\delta>0$ such that, for $0<g\leq g_\delta$, for $s>1/2$, φ, $\psi\in\mathfrak{F}$ and for $n\geq 1$, the limits*

$$\lim_{\epsilon\to 0}(\varphi,\ \langle A^{(\tau)}_n\rangle^{-s}(H-\lambda\pm i\epsilon)\langle A^{(\tau)}_n\rangle^{-s}\psi)$$

exist uniformly for $\lambda\in\Delta_n$. Moreover, for $1/2<s<1$, the map

$$\lambda\mapsto\langle A^{(\tau)}_n\rangle^{-s}(H-\lambda\pm i0)^{-1}\langle A^{(\tau)}_n\rangle^{-s}$$

is Hölder continuous of degree $s-1/2$ in Δ_n.

Here $g_\delta=g^{(3)}_\delta$.

Note that there exists a constant $d_n>0$ such that

$$|a^{(\tau)}_n|^2\leq d_n\langle b\rangle^2 \qquad (7.99)$$

and

$$\left(A^{(\tau)}_n\right)^2\leq d_n P^2 \qquad (7.100)$$

Now,by adapting the proof of theorem 3.3 in [3], we deduce theorem 7.4 from theorem 7.6 and from the following lemma

Lemma 7.7. *Suppose that* $s \in (1/2, 1)$ *and that for some* n, $f \in C_0^\infty(\Delta_n)$. *Then,*

$$\left\| \langle A_n^{(\tau)} \rangle^{-s} \mathrm{e}^{-itH} f(H) \langle A_n^{(\tau)} \rangle^{-s} \right\| = \mathcal{O}\left(t^{-(s-\frac{1}{2})} \right) .$$

We omit the details. □

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(J.-C. Guillot) Centre de Mathématiques Appliquées, UMR 7641, École Polytechnique - C.N.R.S, 91128 Palaiseau Cedex, France

E-mail address: `Jean-Claude.Guillot@polytechnique.edu`